Fourier analytic methods for orthogonal projections

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Joint work with Jonathan Fraser

Shenzhen Technology University - Fractal talk

Motivation

Let $X \subseteq \mathbb{R}^d$ be a Borel set, $e \in S^{d-1}$. Let $P_e : \mathbb{R}^d \to \mathbb{R}$, $P_e(x) = e \cdot x$ be the orthogonal projection map.

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Almost all means that

$$\mathcal{L}^{d-1}\big(\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < \min\{\dim_{\mathsf{H}} X, 1\}\}\big) = 0.$$

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We want to study for $u \in [0, \min\{\dim_{\mathsf{H}} X, 1\}]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\}.$$

A few bounds

Set $X \subseteq \mathbb{R}^2$, $\dim_{\mathsf{H}} X \leqslant 1$ and let $u \in [0, \dim_{\mathsf{H}} X]$.

The first bound by Kaufman '68:

$$\dim_{\mathsf{H}}\{e \in S^1 : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant u,$$

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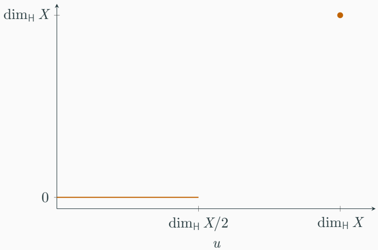

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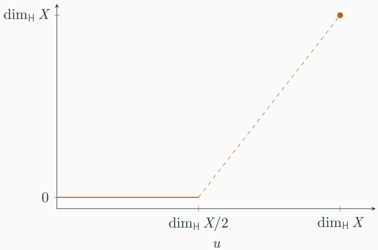

$\min\{0, 2u - \dim_{\mathsf{H}} X\}$
Conjectured by Oberlin

Proved by Ren–Wang '23

The Fourier dimension

Given $X \subseteq \mathbb{R}^d$, recall

$$\dim_{\mathsf{H}} X = \sup\left\{ s \in [0, d] : \exists \mu \text{ finite on } X \text{ with } \int_{\mathbb{R}^d} |\widehat{\mu}(z)|^2 |z|^{s-d}\, dz < \infty \right\};$$

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$$\{e \in S^1 : \dim_{\mathrm{H}} P_e(X) < u\} = \varnothing.$$

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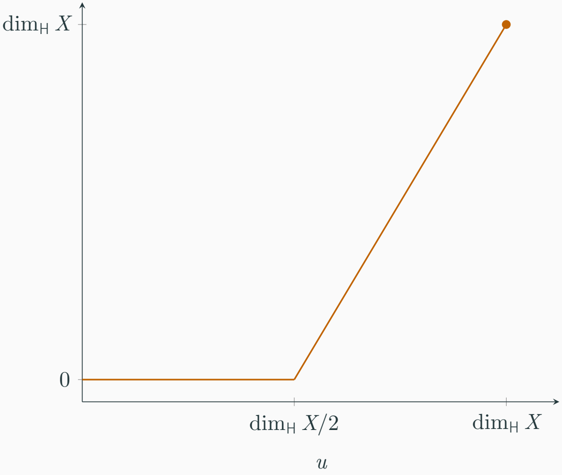

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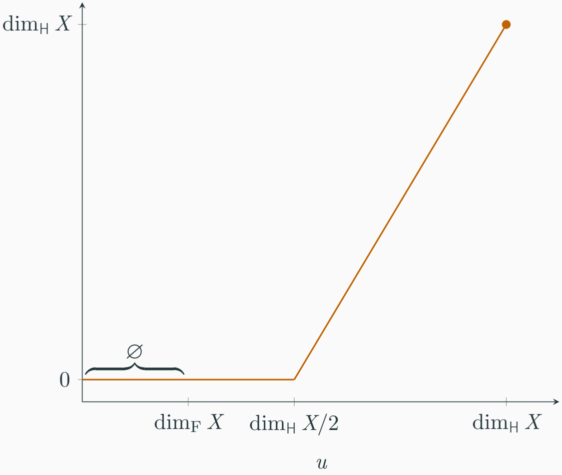

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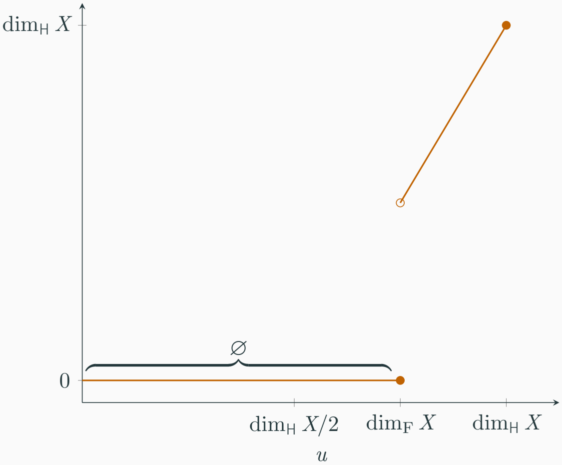

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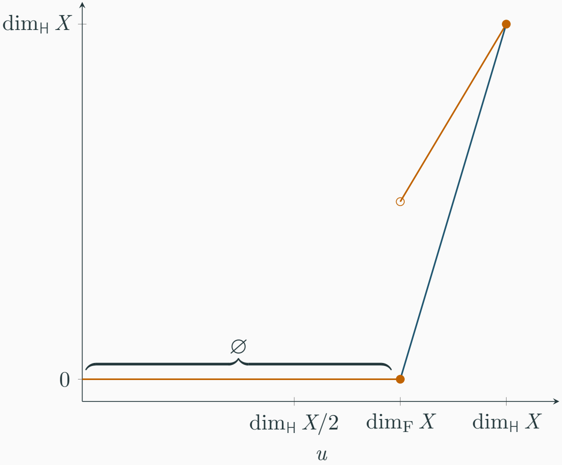

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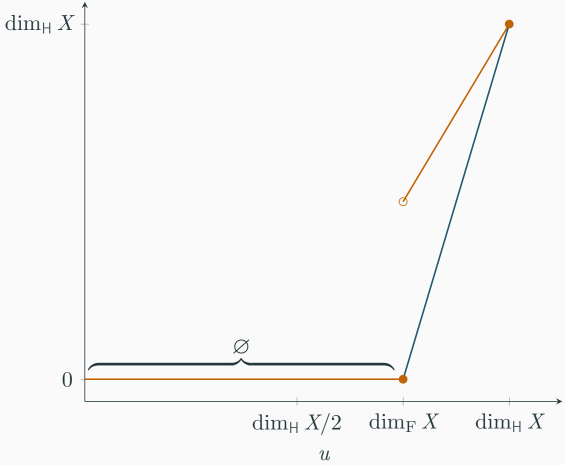

Question

Can we use the Fourier dimension to improve these bounds?

Recall Ren–Wang's bound,

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Example (A 'pointwise sharp' example of RW)

Fix $s \in (0,1]$ *and* $t \in (s/2, s)$. *There exists* $A \subseteq \mathbb{R}^2$ *with* $\dim_{\mathsf{H}} A = s$ *such that* $\dim_{\mathsf{H}}\{e \in S^1 : \dim_{\mathsf{H}} P_e(X) \leqslant t\} = 2t - s$.

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Fix $s \in (0, 1]$ *and* $t \in (s/2, s)$. *Let* A *be the set from the previous example and* B *with* $\dim_{\mathrm{F}} B = \dim_{\mathsf{H}} B = t$. *Then*

- $\dim_{\mathrm{F}}(A \cup B) = t$; $\dim_{\mathsf{H}}(A \cup B) = s$.
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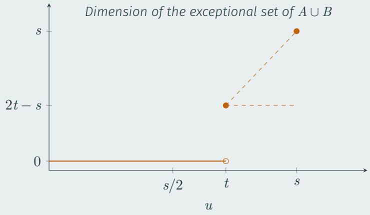

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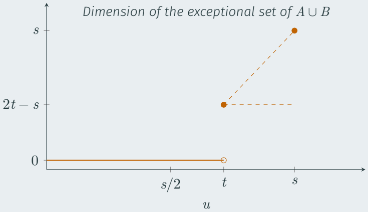

Remark

The Fourier dimension alone is not sufficient to improve the bounds

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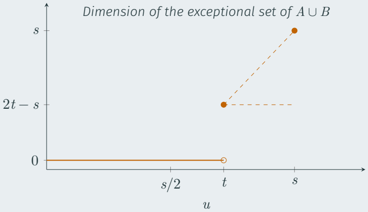

Question

What conditions on Fourier decay give better bounds for the dimension of the exceptional set?

The Fourier spectrum

Given $X \subseteq \mathbb{R}^d$ and $\theta \in (0,1]$

$$\dim_{\mathrm{F}}^{\theta} X = \sup\left\{ s \in [0,d] : \exists \mu \text{ finite on } X \text{ with } \int_{\mathbb{R}^d} \left|\widehat{\mu}(z)\right|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d}\, dz < \infty \right\}.$$

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- $\theta \mapsto \dim_{\mathrm{F}}^{\theta} X$ is continuous and non-decreasing.

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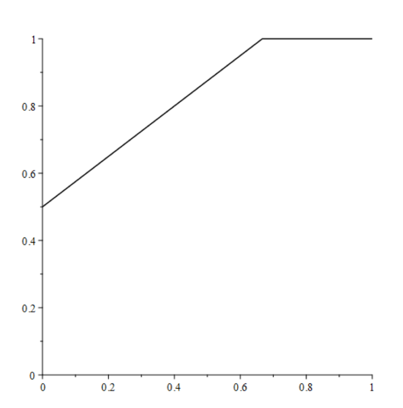

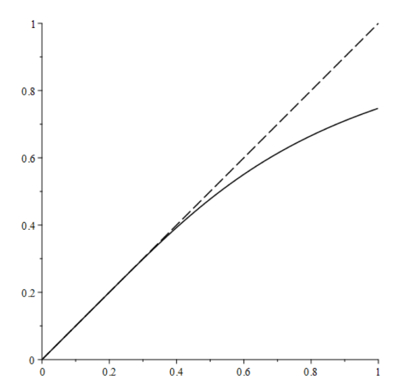

The Fourier spectrum

Given $X \subseteq \mathbb{R}^d$ and $\theta \in (0,1]$

$$\dim_{\mathrm{F}}^{\theta} X = \sup\left\{ s \in [0,d] : \exists \mu \text{ finite on } X \text{ with } \int_{\mathbb{R}^d} \left|\widehat{\mu}(z)\right|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d}\, dz < \infty \right\}.$$

$$\theta = 0 \quad \longrightarrow \quad \sup_{z \in \mathbb{R}^d} \left|\widehat{\mu}(z)\right|^2 |z|^s < \infty$$

Some facts about the Fourier spectrum:

- $\dim_{\mathrm{F}}^{0} X = \dim_{\mathrm{F}} X$ and $\dim_{\mathrm{F}}^{1} X = \dim_{\mathsf{H}} X$.
- For each $\theta \in [0,1]$, $\dim_{\mathrm{F}} X \leqslant \dim_{\mathrm{F}}^{\theta} X \leqslant \dim_{\mathsf{H}} X$.
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Theorem (Fraser–dO, 2024+)

Let $X \subset \mathbb{R}^d$ be a Borel set. If $u \leqslant \sup_{\theta\in[0,1]}(\dim_{\mathrm{F}}^{\theta} X - (d-1)\theta)$, then

$$\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} = \varnothing.$$

Exceptional set estimates

Let $X \subseteq \mathbb{R}^d$, for $u \in [0, \min\{\dim_{\mathsf{H}} X, 1\}]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\}$$

$$\leqslant \begin{cases} 2u - \dim_{\mathsf{H}} X\,, & \text{if } d = 2, \quad \text{(Ren–Wang '23)}; \\ d - 2 + u\,, & \text{if } \dim_{\mathsf{H}} X \leqslant 1, \quad \text{(Mattila '15)}; \\ d - 1 - \dim_{\mathsf{H}} X + u\,, & \text{if } \dim_{\mathsf{H}} X \geqslant 1, \quad \text{(Peres–Schlag '00)}. \end{cases}$$

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Better estimates - $\mathbb{R}^2$

Given $X \subseteq \mathbb{R}^2$, for what $\theta \in [0,1]$ is $1 + \frac{u\text{-}\dim_{\mathrm{F}}^{\theta} X}{\theta} < 2u - \dim_{\mathsf{H}} X$?

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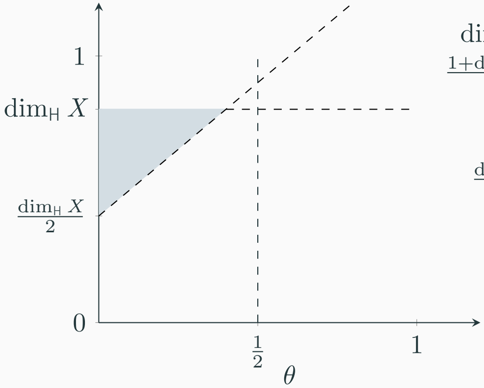

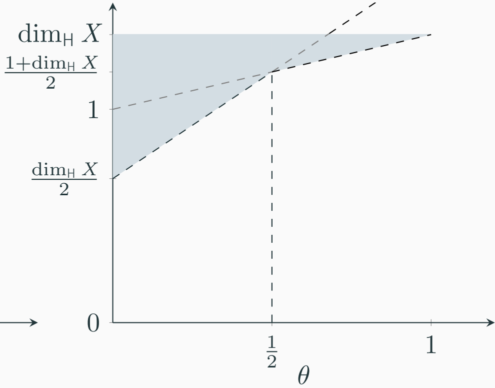

We can improve Ren-Wang's bounds if $\dim_{\mathrm{F}}^{\theta} X$ intersects the shaded region.

Better estimates - Higher dimensions

Given $X \subseteq \mathbb{R}^d$, for what $\theta \in [0,1]$ is $d-1+\frac{u-\dim_{\mathrm{F}}^{\theta} X}{\theta} < d-2+u$ or $d-1+\frac{u-\dim_{\mathrm{F}}^{\theta} X}{\theta} < d-1-\dim_{\mathsf{H}} X + u$?

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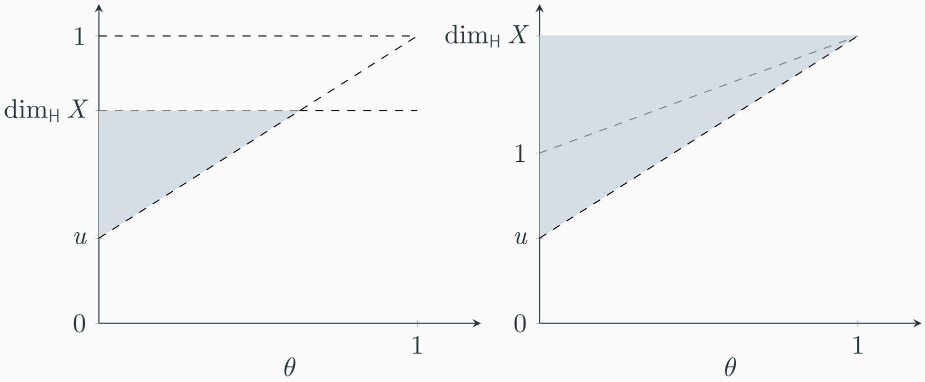
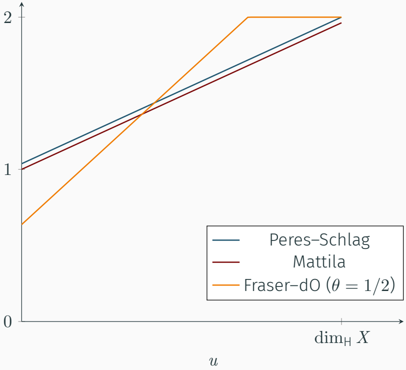


We can improve Mattila's or Peres–Schlag's bounds if $\dim_F^\theta X$ intersects the shaded region.

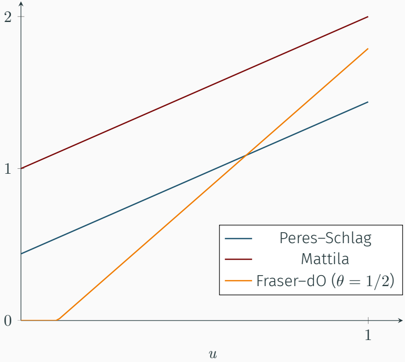
An example on $\mathbb{R}^3$

Let E_α, E_β and E_γ be three middle $(1-2\alpha)$, $(1-2\beta)$ and $(1-2\gamma)$ Cantor sets, respectively. Define $X = E_\alpha \times E_\beta \times E_\gamma$.

$\alpha = 1/5$, $\beta = 1/10$, $\gamma = 1/15$



$\alpha = 1/3$, $\beta = 1/4$, $\gamma = 1/5$



Exceptional set estimates

What more information does

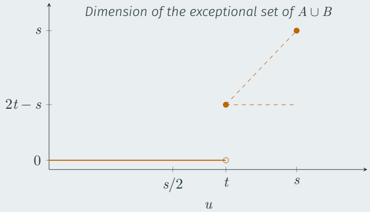
$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant \max\left\{0, d-1+\inf_{\theta\in(0,1]}\frac{u-\dim_{\mathrm{F}}^{\theta} X}{\theta}\right\}$$

give?

Example ($\dim_{\mathrm{F}} X$ gives discontinuous bounds)

Fix $s \in (0,1]$ *and* $t \in (s/2, s)$. *Let* A *be the set from the previous example and* B *with* $\dim_{\mathrm{F}} B = \dim_{\mathrm{H}} B = t$. *Then*

- $\dim_{\mathrm{F}}(A \cup B) = t$; $\dim_{\mathrm{H}}(A \cup B) = s$.
- *If* $u \geqslant t$, $\dim_{\mathrm{H}}\{e \in S^1 : \dim_{\mathrm{H}} P_e(A \cup B) \leqslant u\} \geqslant 2t - s$.



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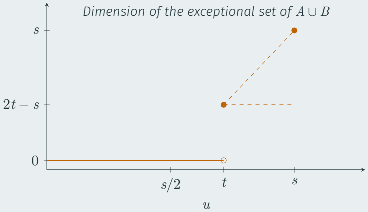

Question

Under what conditions do we get continuity for the bound of the dimension of the exceptional set at $u = \dim_{\mathrm{F}} X$?

Exceptional set estimates

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant \max\left\{0, d-1+\inf_{\theta\in(0,1]} \frac{u-\dim_{\mathrm{F}}^{\theta} X}{\theta}\right\}.$$

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We ask for continuity of the bound at $u = \dim_{\mathrm{F}} X$, let $\varepsilon \in (0,1)$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < \dim_{\mathrm{F}} X + \varepsilon^2\} \leqslant d-1+\inf_{\theta\in(0,1]} \frac{\dim_{\mathrm{F}} X + \varepsilon^2 - \dim_{\mathrm{F}}^{\theta} X}{\theta}$$

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Fix $s \in (0,1]$ and $t \in (s/2, s)$. Let A be the set from the 'pointwise sharp' example and B with $\dim_{\mathrm{F}} B = \dim_{\mathrm{H}} B = t$. Then

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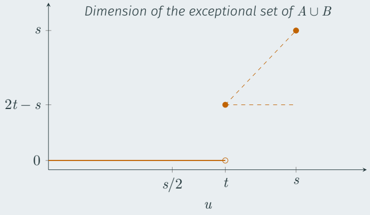

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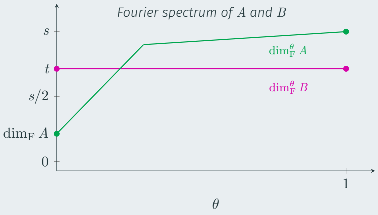

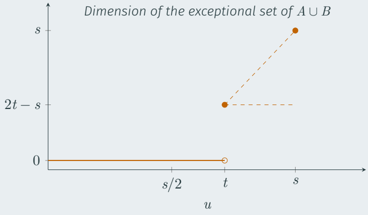

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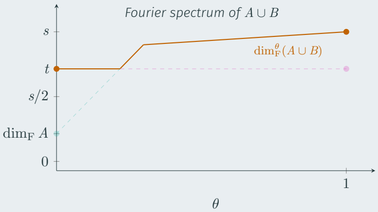

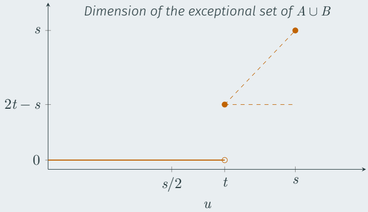

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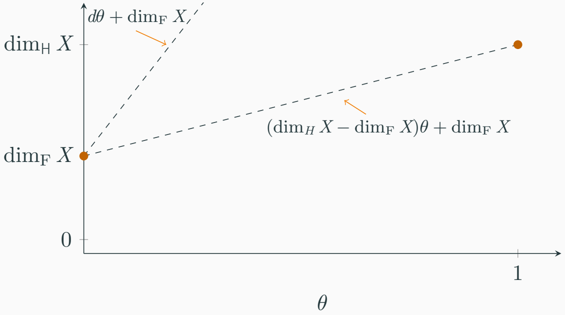

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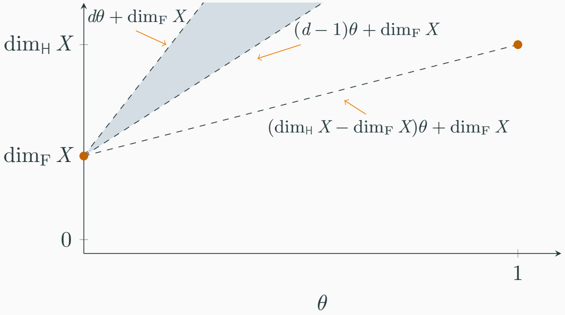

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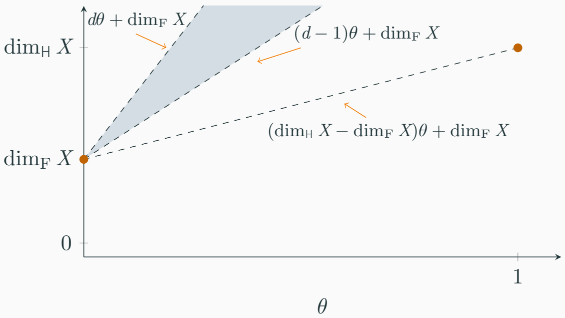

Question

Is $\underline{\partial_+} \dim_{\mathrm{F}}^{\theta} X|_{\theta=0} > 0$ sufficient? Or perhaps $\underline{\partial_+} \dim_{\mathrm{F}}^{\theta} X|_{\theta=0} \geqslant \rho$ for some $\rho > 0$?

Thank you!