Exceptional projections and dimension interpolation

Ana E. de Orellana
University of St Andrews
aedo1@st-andrews.ac.uk

Joint work with Jonathan Fraser

Geometry and Fractals under the Midnight Sun

Motivation

Marstrand's theorem: For any Borel set $X \subseteq \mathbb{R}^d$ and almost all directions $e \in S^{d-1}$, $\dim_{\mathsf{H}} P_e(X) = \min\{\dim_{\mathsf{H}} X, 1\}$.

We are interested in studying the dimension of the exceptional set, for $u \in [0, \min\{\dim_{\mathsf{H}} X, 1\}]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\}$$
$$\leqslant \begin{cases} 2u - \dim_{\mathsf{H}} X, & \text{if } d = 2, \quad \text{(Ren–Wang '23)}; \\ d - 2 + u, & \text{if } \dim_{\mathsf{H}} X \leqslant 1, \quad \text{(Mattila '15)}; \\ d - 1 - \dim_{\mathsf{H}} X + u, & \text{if } \dim_{\mathsf{H}} X \geqslant 1, \quad \text{(Peres–Schlag '00)}. \end{cases}$$

Motivation

Marstrand's theorem: For any Borel set $X \subseteq \mathbb{R}^d$ and almost all directions $e \in S^{d-1}$, $\dim_{\mathsf{H}} P_e(X) = \min\{\dim_{\mathsf{H}} X, 1\}$.

We are interested in studying the dimension of the exceptional set, for $u \in [0, \min\{\dim_{\mathsf{H}} X, 1\}]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant \begin{cases} 2u - \dim_{\mathsf{H}} X, & \text{if } d = 2, \quad \text{(Ren–Wang '23)}; \\ d - 2 + u, & \text{if } \dim_{\mathsf{H}} X \leqslant 1, \quad \text{(Mattila '15)}; \\ d - 1 - \dim_{\mathsf{H}} X + u, & \text{if } \dim_{\mathsf{H}} X \geqslant 1, \quad \text{(Peres–Schlag '00)}. \end{cases}$$

If $X \subseteq \mathbb{R}^d$ is a Salem set ($\dim_{\mathrm{F}} X = \dim_{\mathsf{H}} X$), then there are no exceptions to Marstrand's theorem.

Motivation

Marstrand's theorem: For any Borel set $X \subseteq \mathbb{R}^d$ and almost all directions $e \in S^{d-1}$, $\dim_{\mathsf{H}} P_e(X) = \min\{\dim_{\mathsf{H}} X, 1\}$.

We are interested in studying the dimension of the exceptional set, for $u \in [0, \min\{\dim_{\mathsf{H}} X, 1\}]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant \begin{cases} 2u - \dim_{\mathsf{H}} X, & \text{if } d = 2, \quad \text{(Ren–Wang '23)}; \\ d - 2 + u, & \text{if } \dim_{\mathsf{H}} X \leqslant 1, \quad \text{(Mattila '15)}; \\ d - 1 - \dim_{\mathsf{H}} X + u, & \text{if } \dim_{\mathsf{H}} X \geqslant 1, \quad \text{(Peres–Schlag '00)}. \end{cases}$$

If $X \subseteq \mathbb{R}^d$ is a Salem set $(\dim_{\mathrm{F}} X = \dim_{\mathsf{H}} X)$, then there are no exceptions to Marstrand's theorem.

Can we use Fourier decay to give better estimates?

The Fourier spectrum

Given a Borel set $X \subseteq \mathbb{R}^d$, we know that

$$\dim_{\mathrm{F}} X \leqslant \dim_{\mathsf{H}} X,$$

The Fourier spectrum

Given a Borel set $X \subseteq \mathbb{R}^d$, the Fourier spectrum of X at $\theta \in [0,1]$ interpolates between the two

$$\dim_{\mathrm{F}} X \leqslant \dim_{\mathrm{F}}^{\theta} X \leqslant \dim_{\mathsf{H}} X,$$

We have that, $\dim_{\mathrm{F}}^{0} X = \dim_{\mathrm{F}} X$ and $\dim_{\mathrm{F}}^{1} X = \dim_{\mathsf{H}} X$.

The Fourier spectrum

Given a Borel set $X \subseteq \mathbb{R}^d$, the Fourier spectrum of X at $\theta \in [0,1]$ interpolates between the two

$$\dim_{\mathrm{F}} X \leqslant \dim_{\mathrm{F}}^{\theta} X \leqslant \dim_{\mathsf{H}} X,$$

We have that, $\dim_{\mathrm{F}}^{0} X = \dim_{\mathrm{F}} X$ and $\dim_{\mathrm{F}}^{1} X = \dim_{\mathsf{H}} X$.

Theorem (Fraser–dO, 2024+)

Let $X \subseteq \mathbb{R}^d$ be a Borel set and $\theta \in (0,1]$. Then for all $u \in [0,1]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant \max\left\{0, d-1+ \inf_{\theta \in (0,1]} \frac{u - \dim_{\mathrm{F}}^{\theta} X}{\theta}\right\}.$$

The Fourier spectrum

Given a Borel set $X \subseteq \mathbb{R}^d$, the Fourier spectrum of X at $\theta \in [0,1]$ interpolates between the two

$$\dim_{\mathrm{F}} X \leqslant \dim_{\mathrm{F}}^{\theta} X \leqslant \dim_{\mathsf{H}} X,$$

We have that, $\dim_{\mathrm{F}}^{0} X = \dim_{\mathrm{F}} X$ and $\dim_{\mathrm{F}}^{1} X = \dim_{\mathsf{H}} X$.

Theorem (Fraser–dO, 2024+)

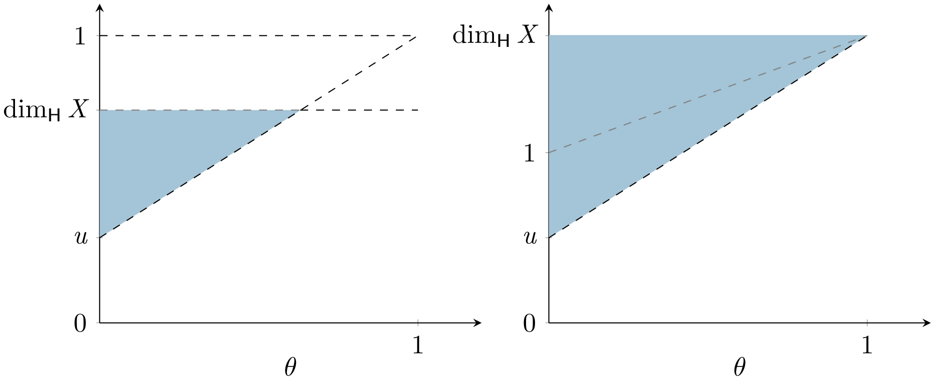
Let $X \subseteq \mathbb{R}^d$ be a Borel set and $\theta \in (0,1]$. Then for all $u \in [0,1]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant \max\left\{0, d-1+\inf_{\theta \in (0,1]} \frac{u - \dim_{\mathrm{F}}^{\theta} X}{\theta}\right\}.$$

Is this bound any good?

Getting better estimates - High dimensions

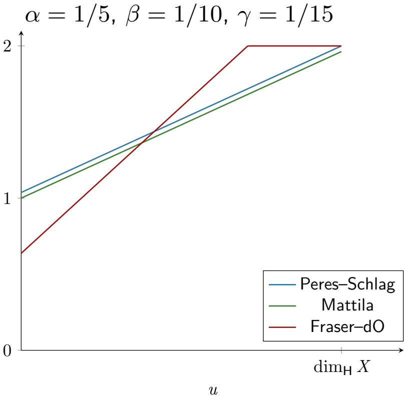
If $X \subseteq \mathbb{R}^d$ with $d \geqslant 3$

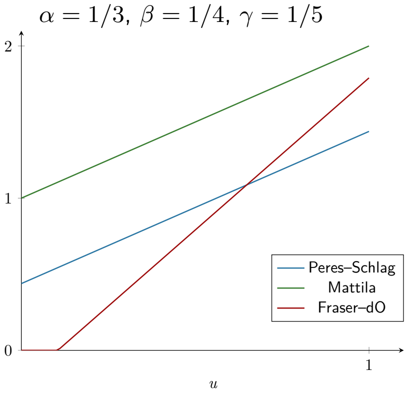


We can improve Mattila's or Peres–Schlag's bounds if $\dim_{\mathrm{F}}^{\theta} X$ intersects the shaded region.

A concrete example

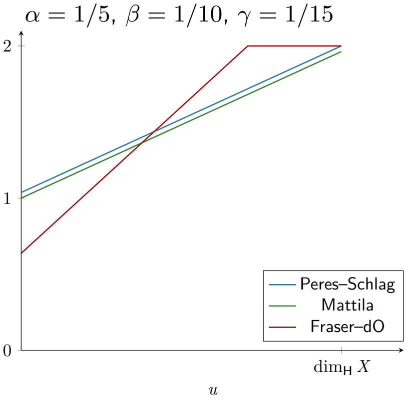
Let E_α, E_β and E_γ be three middle $(1-2\alpha), (1-2\beta)$ and $(1-2\gamma)$ Cantor sets, respectively. Define $X = E_\alpha \times E_\beta \times E_\gamma$.

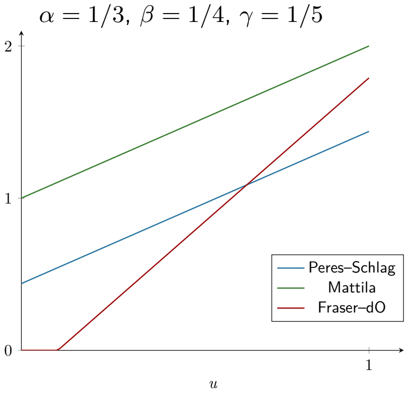




A concrete example

Let E_α, E_β and E_γ be three middle $(1-2\alpha),(1-2\beta)$ and $(1-2\gamma)$ Cantor sets, respectively. Define $X = E_\alpha \times E_\beta \times E_\gamma$.





However, improvement is possible for a larger family of sets satisfying a mild non-concentration condition.

Thank you!