Multifractal measures and the Fourier restriction problem

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Fractals and Related Fields V

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The Fourier restriction/extension problem

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- Posed by Stein in 1960s.
- Applications to PDEs.

General measures

Mockenhaupt (2000), Mitsis (2002), and Bak–Seeger (2011):

Theorem (Stein–Tomas Theorem)

Let μ *be a non-zero, finite, compactly supported, Borel measure on* $\mathbb{R}^d$ *with*

$$\mu(B(x,r)) \lesssim r^{\alpha}, \quad \forall x \in \mathbb{R}^d, r > 0;$$
$$\sup_{\xi \in \mathbb{R}^d} \left|\widehat{\mu}(\xi)\right|^2 |\xi|^{\beta} < \infty.$$

If

$$p \geqslant 2 + \frac{4(d-\alpha)}{\beta},$$

then $\|\widehat{f\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$

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$$D_\mu(\infty) = \sup\left\{\alpha : \mu(B(x,r)) \lesssim r^\alpha, \quad \forall x \in \mathbb{R}^d, r > 0\right\};$$

$$\dim_{\mathrm{F}} \mu = \sup\left\{\beta : \sup_{\xi \in \mathbb{R}^d} \left|\widehat{\mu}(\xi)\right|^2 |\xi|^\beta < \infty\right\}.$$

If

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Can more information about the dimensionality of μ give us a better range for the restriction estimate to hold?

The L^2 dimension is defined as

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When is this range better than Stein–Tomas'?

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When is

$$D_\mu(\infty) + \frac{1}{2}\dim_{\mathrm{F}} \mu < D_\mu(2)$$

Multifractal measures

Multifractal measures distributed in a highly irregular way.

Natural middle third Cantor measure:

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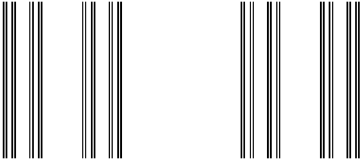

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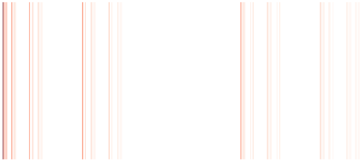

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- What are $D_\mu(\infty)$ and $D_\mu(2)$?

The L^q-dimensions

For $q > 1$.

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$$D_\mu(0) = \underline{\dim}_{\mathrm{B}} \operatorname{spt} \mu; \qquad \lim_{q \searrow 1} D_\mu(q) \leqslant \dim_{\mathrm{H}} \mu;$$

$$D_\mu(2) \leqslant \dim_{\mathrm{H}} \operatorname{spt} \mu; \qquad \lim_{q \to \infty} D_\mu(q) = D_\mu(\infty).$$

Restriction theorems for the L^q-dimensions

Theorem (Carnovale-Fraser-dO, 2026+)

Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If

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- $q \to \infty$: $p > 2 + \dfrac{4(d - D_\mu(\infty))}{\dim_{\mathrm{F}} \mu}$. Recovers Stein–Tomas.
- $q = 2$: $p > 4 + \dfrac{4(d - D_\mu(2))}{\dim_{\mathrm{F}} \mu}$. Recovers CFdO24.

Mandelbrot cascade measure

Divide the unit square into N equal squares and assign to each a random weight given by W with $\mathbb{E}(W) = 1$. Then iterate.

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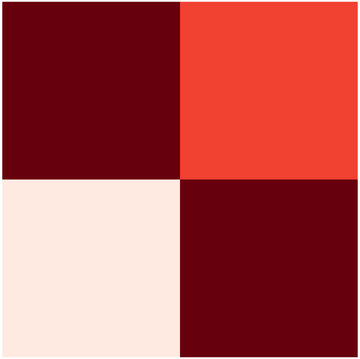

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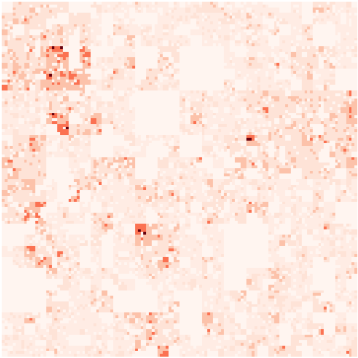

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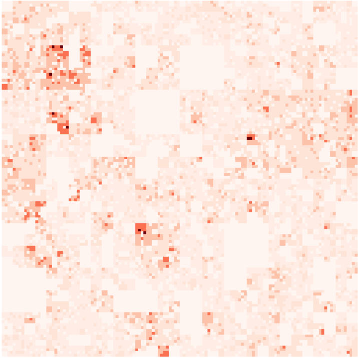

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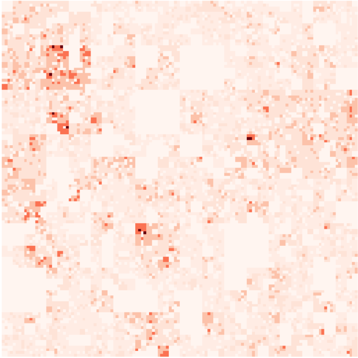

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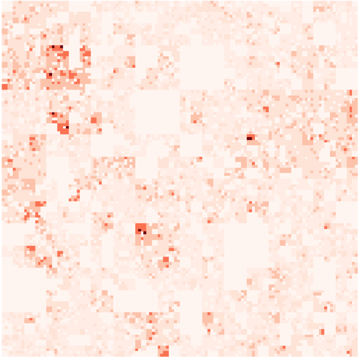
$$W = \begin{cases} 2 & \text{with probability } 0.1 \\ 1 & \text{with probability } 0.8 \\ 0 & \text{with probability } 0.1 \end{cases}$$



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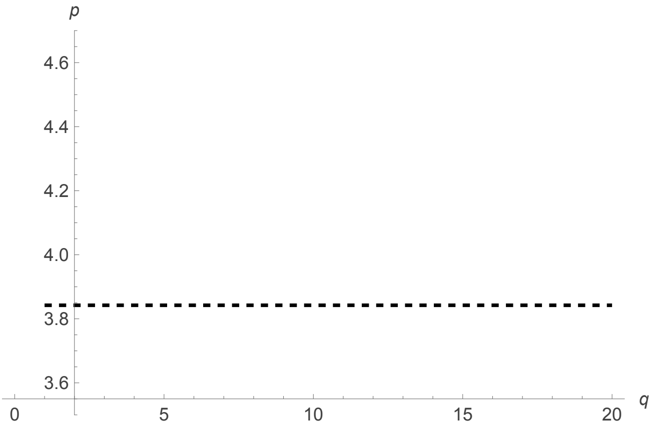

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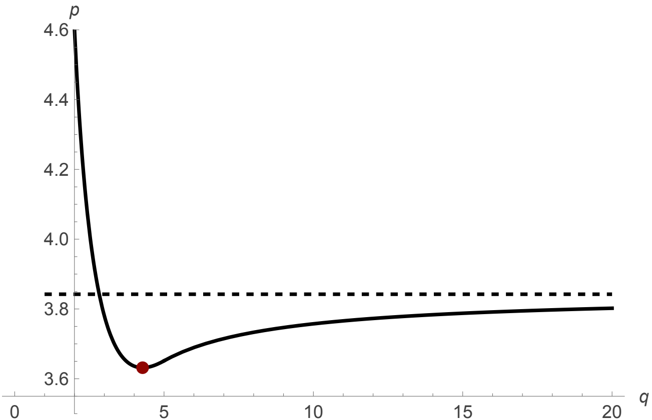

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CFdO26:

$$p > \frac{2q}{q-1} + \frac{4(d - D_\mu(q))}{\dim_{\mathrm{F}} \mu} \geqslant 3.63$$

Merci!