The continuity of exceptional estimates for orthogonal projections

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Joint work with Jonathan Fraser

Fractal Geometry and Stochastics 7

Motivation

Theorem (Marstrand's projection theorem)

For any Borel set $X \subseteq \mathbb{R}^d$ *and almost all directions* $e \in S^{d-1}$, $\dim_{\mathsf{H}} P_e(X) = \min\{\dim_{\mathsf{H}} X, 1\}$.

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$$\mathcal{L}^{d-1}\big(\big\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < \min\{\dim_{\mathsf{H}} X, 1\}\big\}\big) = 0.$$

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We want to study for $u \in [0, \min\{\dim_{\mathsf{H}} X, 1\}]$,

$$\dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\}.$$

A few bounds

Set $X \subseteq \mathbb{R}^2$, $\dim_{\mathsf{H}} X \leqslant 1$ and let $u \in [0, \dim_{\mathsf{H}} X]$.

The first bound by Kaufman '68:

$$\dim_{\mathsf{H}}\{e \in S^1 : \dim_{\mathsf{H}} P_e(X) < u\} \leqslant u,$$

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$$\dim_{\mathsf{H}}\{e \in S^1 : \dim_{\mathsf{H}} P_e(X) < \dim_{\mathsf{H}} X/2\} = 0.$$

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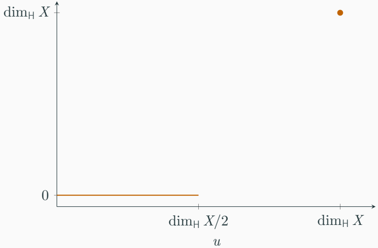

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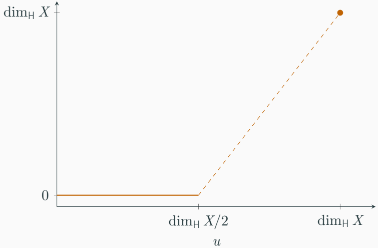

$\min\{0, 2u - \dim_H X\}$
Conjectured by Oberlin

Proved by Ren–Wang '23

The Fourier dimension

Given $X \subseteq \mathbb{R}^d$, recall

$$\dim_{\mathsf{H}} X = \sup\left\{ s : \exists\mu \text{ finite on } X \text{ with } \int_{\mathbb{R}^d} |\widehat{\mu}(z)|^2 |z|^{s-d}\, dz < \infty \right\};$$

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Thus, if $u \leqslant \dim_{\mathrm{F}} X$,

$$\{e \in S^1 : \dim_{\mathsf{H}} P_e(X) < u\} = \varnothing.$$

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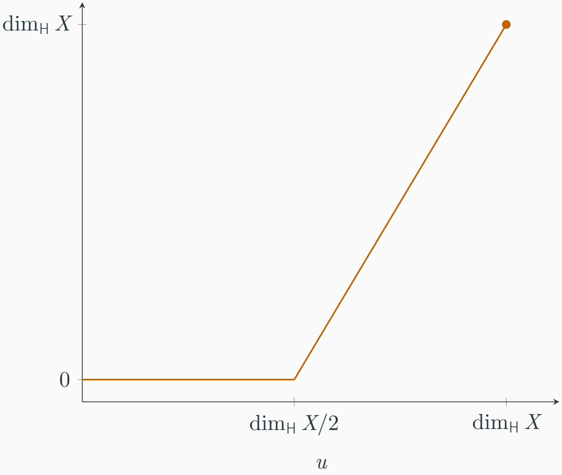

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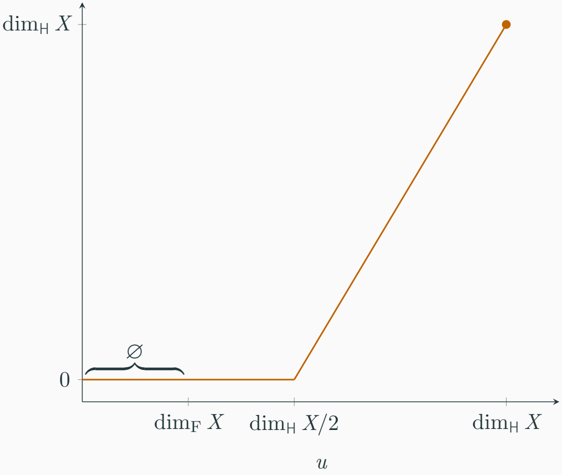

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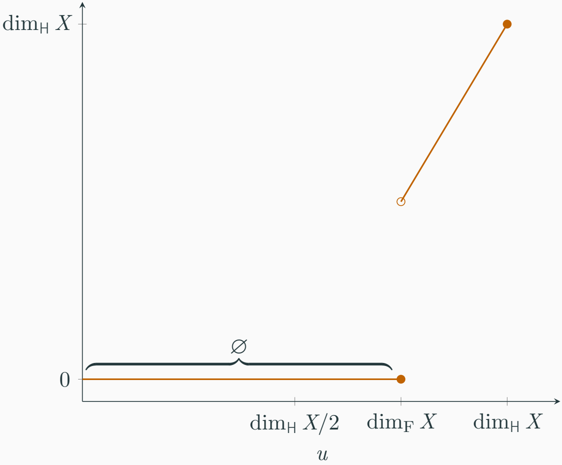

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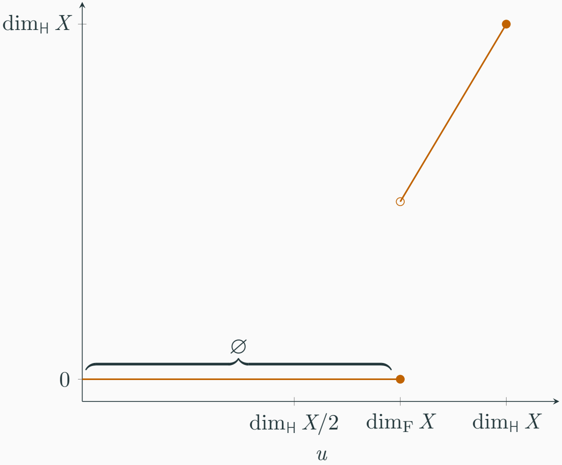

Question

Are there examples where this discontinuity occurs?

Recall Ren–Wang's bound,

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Example (A 'pointwise sharp' example of RW)

Fix $s \in (0,1]$ *and* $t \in (s/2, s)$. *There exists* $A \subseteq \mathbb{R}^2$ *with* $\dim_{\mathsf{H}} A = s$ *such that* $\dim_{\mathsf{H}}\{e \in S^1 : \dim_{\mathsf{H}} P_e(X) \leqslant t\} = 2t - s$.

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Fix $s \in (0,1]$ *and* $t \in (s/2, s)$. *Let* A *be the set from the previous example and* B *with* $\dim_{\mathrm{F}} B = \dim_{\mathsf{H}} B = t$. *Then*

- $\dim_{\mathrm{F}}(A \cup B) = t$; $\dim_{\mathsf{H}}(A \cup B) = s$.
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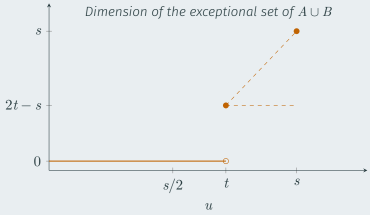

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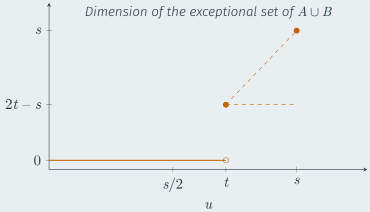

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What conditions on Fourier decay give continuity for the bound of the dimension of the exceptional set at $u = \dim_{\mathrm{F}} X$?

The Fourier spectrum

Given $X \subseteq \mathbb{R}^d$ and $\theta \in (0,1]$

$$\dim_{\mathrm{F}}^{\theta} X = \sup\left\{ s \in [0,d] : \exists \mu \text{ finite on } X \text{ with } \int_{\mathbb{R}^d} \left|\widehat{\mu}(z)\right|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d}\, dz < \infty \right\}.$$

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Exceptional set estimates

Theorem (Fraser–dO, 2024+)

Let $X \subseteq \mathbb{R}^d$ be a Borel set. Then for all $u \in [0,1]$,

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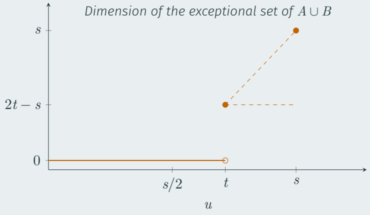

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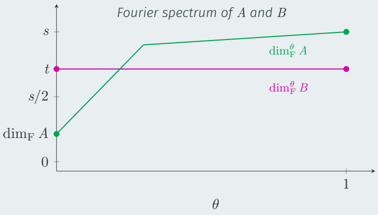

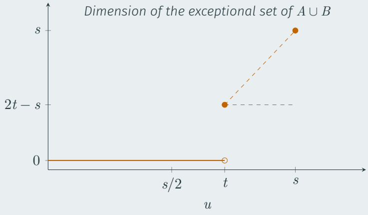

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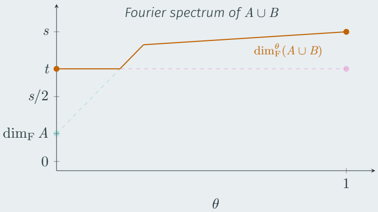

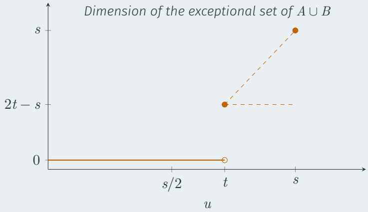

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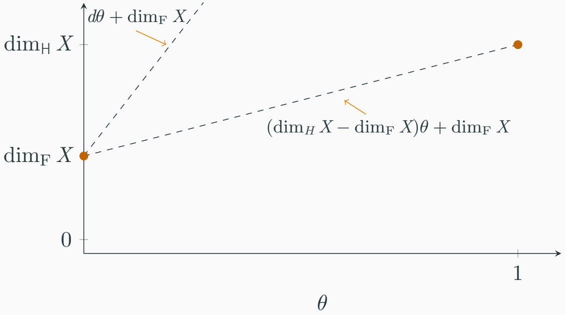

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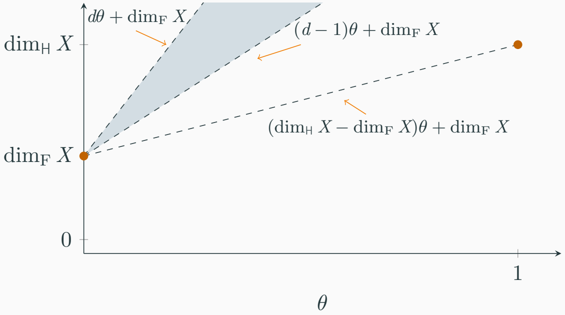
Let X be a Borel set in $\mathbb{R}^d$. If $\underline{\partial_+} \dim_{\mathrm{F}}^{\theta} X|_{\theta=0} \geqslant d-1$, then the function $u \mapsto \dim_{\mathsf{H}}\{e \in S^{d-1} : \dim_{\mathsf{H}} P_e(X) < u\}$ is continuous at $u = \dim_{\mathrm{F}} X$.



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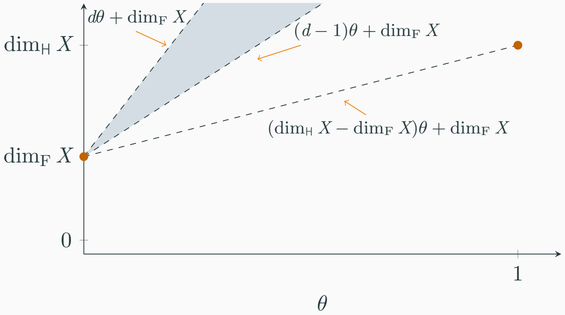

Question

Is $\underline{\partial_+} \dim_{\mathrm{F}}^{\theta} X|_{\theta=0} > 0$ sufficient? Or perhaps $\underline{\partial_+} \dim_{\mathrm{F}}^{\theta} X|_{\theta=0} \geqslant \rho$ for some $\rho > 0$?

Danke schön!