# Projection theorems and the Fourier spectrum

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Joint work with Jonathan Fraser

Workshop on the Geometry of Deterministic and Random Fractals II

# Motivation

**Theorem (Marstrand projection theorem)**

*For any Borel set* $X \subseteq \mathbb{R}^d$ *and almost all directions* $e \in S^{d-1}$,
$\dim_{\mathrm{H}} P_e(X) = \min\{\dim_{\mathrm{H}} X, 1\}$.

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$$\mathcal{L}^{d-1}\big(\big\{ e \in S^{d-1} : \dim_{\mathrm{H}} P_e(X) < \min\{\dim_{\mathrm{H}} X, 1\}\big\}\big) = 0.$$

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We want to study for $u \in [0, \min\{\dim_{\mathrm{H}} X, 1\}]$,

$$\dim_{\mathrm{H}}\{e \in S^{d-1} : \dim_{\mathrm{H}} P_e(X) < u\}.$$

# A few bounds

Set $X \subseteq \mathbb{R}^2$, $\dim_{\mathrm{H}} X \leqslant 1$ and let $u \in [0, \dim_{\mathrm{H}} X]$.

The first bound by Kauffman '68:

$$\dim_{\mathrm{H}}\{e \in S^1 : \dim_{\mathrm{H}} P_e(X) < u\} \leqslant u,$$

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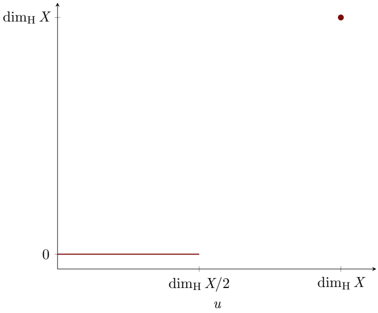

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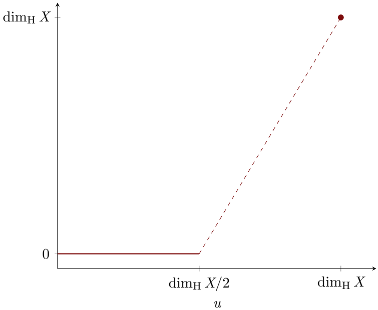


$\min\{0, 2u - \dim_{\mathrm{H}} X\}$
Conjectured by Oberlin

Proved by Ren–Wang '23

# The Fourier dimension

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$$\dim_{\mathrm{F}} X = \sup \Big\{ s \in [0, d] : \exists \mu \text{ finite on } X \text{ with } \sup_z \big|\widehat{\mu}(z)\big|^2 |z|^s < \infty \Big\}.$$

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More generally, if $u \leqslant \dim_{\mathrm{F}} X$,

$$\{e \in S^{d-1} : \dim_{\mathrm{H}} P_e(X) < u\} = \varnothing.$$

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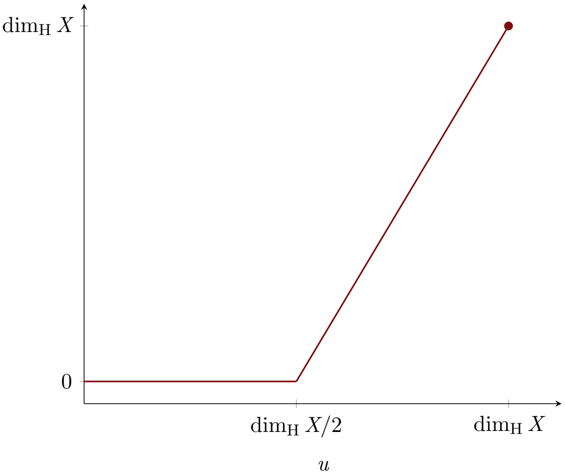

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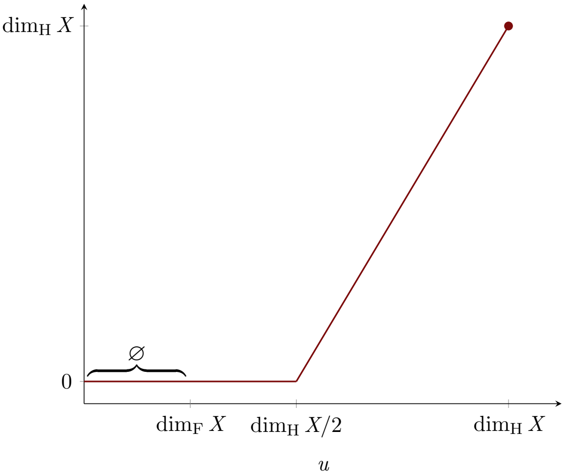

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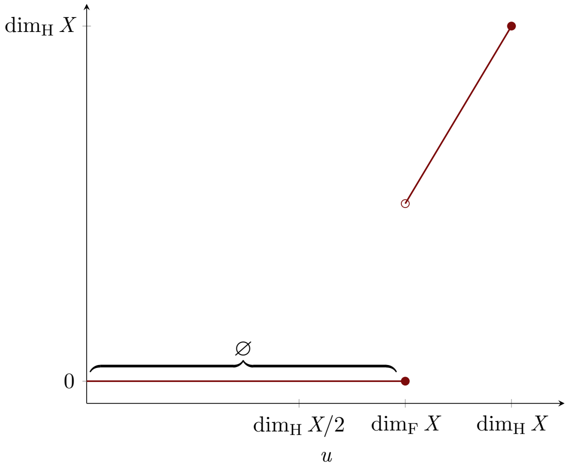

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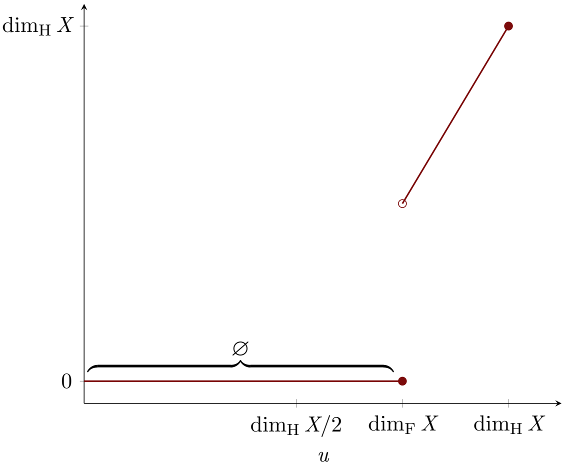


**Question**

*Can we use Fourier decay to get better estimates?*

# The Fourier spectrum

Given $X \subseteq \mathbb{R}^d$ and $\theta \in (0,1]$

$$\dim_{\mathrm{F}}^{\theta} X = \sup\left\{ s \in [0,d] : \exists \mu \text{ finite on } X \text{ with } \int_{\mathbb{R}^d} \left|\widehat{\mu}(z)\right|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d}\, dz < \infty \right\}.$$

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Given $X \subseteq \mathbb{R}^d$ and $\theta \in [0,1]$

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Some facts about the Fourier spectrum:

- $\theta \mapsto \dim_{\mathrm{F}}^{\theta} X$ is continuous and non-decreasing.
- For each $\theta \in [0,1]$, $\dim_{\mathrm{F}} X \leqslant \dim_{\mathrm{F}}^{\theta} X \leqslant \dim_{\mathrm{H}} X$.
- $\dim_{\mathrm{F}}^{0} X = \dim_{\mathrm{F}} X$ and $\dim_{\mathrm{F}}^{1} X = \dim_{\mathrm{H}} X$.

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- For almost all $e \in S^{d-1}$, $\dim_{\mathrm{F}}^{\theta} P_e(X) \geqslant \min\{1, \dim_{\mathrm{F}}^{\theta} X\}$.
- For all $e \in S^{d-1}$, $\dim_{\mathrm{F}}^{\theta} P_e(X) \geqslant \min\{1, \dim_{\mathrm{F}}^{\theta} X - (d-1)\theta\}$.

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- For all $e \in S^{d-1}$, $\dim_{\mathrm{F}}^{\theta} P_e(X) \geqslant \min\{1, \dim_{\mathrm{F}}^{\theta} X - (d-1)\theta\}$.

**Theorem (Fraser–dO, 2024+)**

*Let $X \subset \mathbb{R}^d$ be a Borel set. If $u \leqslant \sup_{\theta \in [0,1]}(\dim_{\mathrm{F}}^{\theta} X - (d-1)\theta)$, then*

$$\{e : S^{d-1} : \dim_{\mathrm{H}} P_e(X) < u\} = \varnothing.$$

# Exceptional set estimates

Let $X \subseteq \mathbb{R}^d$, for $u \in [0, \min\{\dim_{\mathrm{H}} X, 1\}]$,

$$\dim_{\mathrm{H}}\{e \in S^{d-1} : \dim_{\mathrm{H}} P_e(X) < u\}$$
$$\leqslant \begin{cases} 2u - \dim_{\mathrm{H}} X, & \text{if } d = 2, \quad \text{(Ren–Wang '23)}; \\ d - 2 + u, & \text{if } \dim_{\mathrm{H}} X \leqslant 1, \quad \text{(Mattila '15)}; \\ d - 1 - \dim_{\mathrm{H}} X + u, & \text{if } \dim_{\mathrm{H}} X \geqslant 1, \quad \text{(Peres–Schlag '00)}. \end{cases}$$

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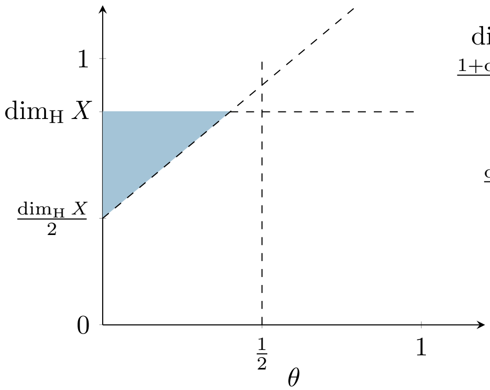
**Theorem (Fraser–dO, 2024+)**

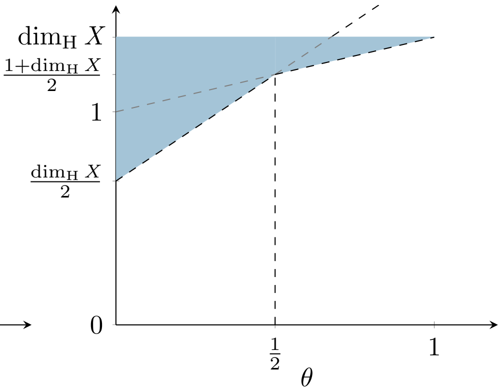
*Let $X \subseteq \mathbb{R}^d$ be a Borel set. Then for all $u \in [0, 1]$,*

$$\dim_{\mathrm{H}}\{e \in S^{d-1} : \dim_{\mathrm{H}} P_e(X) < u\} \leqslant \max\left\{0,\, d-1+\inf_{\theta\in(0,1]} \frac{u - \dim_{\mathrm{F}}^{\theta} X}{\theta}\right\}.$$

# Better estimates - $\mathbb{R}^2$

If $X \subseteq \mathbb{R}^2$

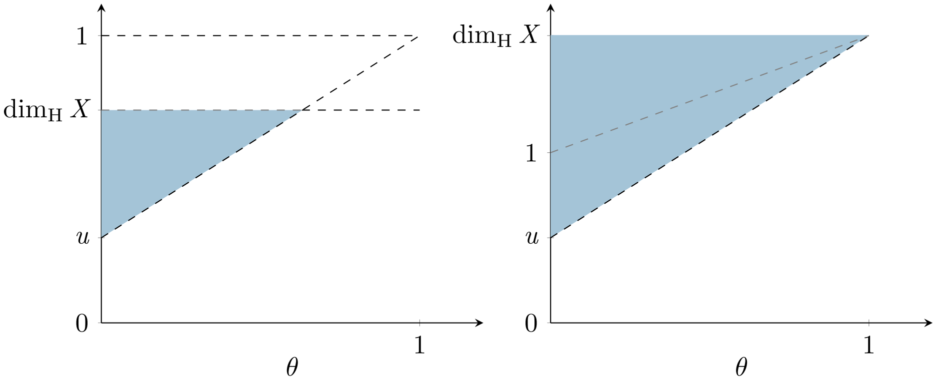



We can improve Ren–Wang's bounds if $\dim_F^\theta X$ intersects the shaded region.

# Better estimates - Higher dimensions

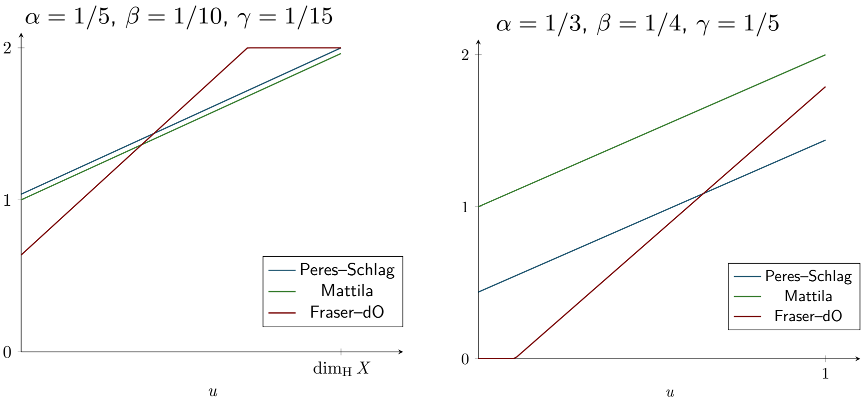
If $X \subseteq \mathbb{R}^d$ with $d \geqslant 3$




We can improve Mattila's or Peres–Schlag's bounds if $\dim_{\mathrm{F}}^{\theta} X$ intersects the shaded region.

# An example on $\mathbb{R}^3$

Let $E_\alpha$, $E_\beta$ and $E_\gamma$ be three middle $(1-2\alpha),(1-2\beta)$ and $(1-2\gamma)$ Cantor sets, respectively. Define $X = E_\alpha \times E_\beta \times E_\gamma$.



**Köszönöm!**

(Thank you!)