

NEW PERSPECTIVES IN HARMONIC ANALYSIS AND FRACTAL GEOMETRY

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Abstract

This Thesis aims to develop a deeper connection between Harmonic Analysis and Fractal geometry. The work in this Thesis is based on the following research papers I wrote during my time as a PhD student: [15, 39, 14, 40, 16, 91].

The first part of this Thesis considers the study of orthogonal projections. Whenever a set supports a measure with maximal polynomial Fourier decay pointwise, all directions satisfy the conclusion of Marstrand’s theorem. For this reason, we study how the average Fourier decay of measures supported on fractal sets affects the Hausdorff dimension of their exceptional set for orthogonal projections. We obtain a new exceptional set estimate for fractal sets depending on the average Fourier decay of measures they support, which enables us to understand for which fractal sets the best-known bounds are suboptimal, thereby improving on classical results in certain cases.

The rest of the Thesis is concerned with the Fourier restriction problem on fractal sets. We generalise the Stein–Tomas theorem in two ways. First, by considering the average Fourier decay of fractal measures, we show that this improves known estimates for many examples, such as multifractal measures and the graph of additive Brownian motion. Second, we show that further improvement can be achieved by considering the L^q -dimensions, widely used to capture the multifractal behaviour of measures. We then apply this estimate to the Mandelbrot cascade measure. Finally, we consider multilinear Fourier extension estimates, and find sufficient and necessary conditions that do not always coincide and depend on whether the convolution of the measures at play is absolutely continuous or singular. The positive estimate depends on the smoothness of the convolution of the measures, by requiring it to be absolutely continuous and lying in some Lebesgue space. To obtain the necessary conditions, we build random measures that contain transversal ‘flat’ structures within them.

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Publications and Collaborations

I would first like to thank my collaborators during my PhD for many engaging and enlightening conversations: Jonathan Fraser, Marc Carnovale, and Itamar Oliveira. I have learned a lot from all of them. This Thesis contains work which has been adapted from the following publications:

- J. M. Fraser and A. E. de Orellana. A Fourier analytic approach to exceptional set estimates for orthogonal projections.
To appear in *Indiana Univ. Math. J.*, <https://arxiv.org/abs/2404.11179>
- M. Carnovale, J. M. Fraser, and A. E. de Orellana. Obtaining the Fourier spectrum via Fourier coefficients.
To appear in *Proc. Amer. Math.*, <https://arxiv.org/abs/2403.12603>
- M. Carnovale, J. M. Fraser, and A. E. de Orellana. L^2 restriction estimates from the Fourier spectrum.
Preprint (submitted), <https://arxiv.org/abs/2412.14896>
- J. M. Fraser and A. E. de Orellana. Fourier restriction for the additive Brownian sheet.
Preprint (submitted), <https://arxiv.org/abs/2601.05802>
- I. Oliveira and A. E. de Orellana. Multiple convolutions and multilinear fractal Fourier extension estimates.
Preprint (submitted), <https://arxiv.org/abs/2602.08568>

In addition, Chapter 5 contains material from the following unpublished paper.

- M. Carnovale, J. M. Fraser, and A. E. de Orellana. Fourier restriction estimates based on L^q -dimensions: beyond Stein–Tomas.

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Declarations

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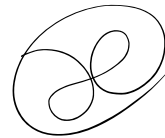
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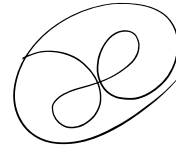
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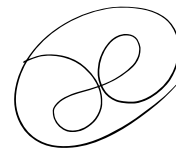
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CHAPTER 1

Fractal Geometry and Dimension Theory

1.1. Fractal Geometry

The basic act of measuring, assigning numbers to physical quantities, is part of an innate human desire to understand and organise the world around us. It is so fundamental that its history can be traced back to as far as the ancient Egyptians around 3000 BC, who first mastered the measurement of time with the Egyptian sundial during the day, and the water clock at night. The Egyptians used the cubit as their standard unit of *length* and used it to incredible precision, for example, the Great Pyramid of Giza has a 440 cubits long base, and it is 280 cubits high, exactly. In fact, Egyptians were the first civilisation to standardise a unit, and they did this by comparing their wooden rods against the royal granite reference; failure to maintain the standard was punished with death. This harsh way of implementing the standard unit meant that they were able to measure things with more precision than ever before, and it was the key to the success of the Egyptian empire.

Each year, the Nile's floods erased all land boundaries, forcing the Egyptians to redistribute their fields. For this reason, and to divide the land fairly, fields would always have the shape of simple figures such as rectangles and triangles, whose *areas* could be easily calculated. Far from reflecting a lack of sophistication, this emphasis on simple geometric forms demonstrates the Egyptians' mastery of practical mathematics and engineering. Their monumental architecture often followed the same principles, with pyramids and other structures adopting simple shapes. One of the most advanced discoveries of ancient Egyptian geometry was the calculation of the *volume* of an incomplete pyramid, a frustum, which allowed them to know exactly how much additional material was necessary to complete the structure.

Implicit in all these practices is the awareness that length is one-dimensional, area is two-dimensional, and volume is three-dimensional. The ancient Greeks continued this development with Euclidean geometry, using structures such as lines, circles and polygons to idealise and describe human-scaled observations. Yet, the world is full of irregular shapes, like coastlines, trees, and ferns that are too irregular to be one-dimensional lines, and too insubstantial to be two-dimensional surfaces or three-dimensional solids. For these shapes, the standard notions of length, area, and volume



FIGURE 1. Left: The coast of St Andrews and its paradoxical length. Right: An autumnal tree at Cambo Gardens in Kingsbarns.

no longer work. Although fractal geometry first emerged to use these irregularities as counterexamples in calculus, it soon became another iteration of humanity's effort to describe what is around us, this time with the objective of describing things to infinite detail.

When considering fractal sets, we are interested in understanding how they 'fill up' space by measuring their complexity, or structure; to this end, we assign to them a number called *dimension*. Dimension theory concerns the description and classification of fractal objects. For this purpose, a single fractal set will have several dimensions assigned to it, each more or less useful depending on what we want to measure on the set. The most natural way of defining a dimension of a set is in terms of growth rates of covers of the set. If we repeatedly divide a line segment into halves, each division doubles the number of pieces. A straight line will be composed of 2 pieces when halved, 4 when quartered, and $(2^k)^1$ pieces when divided into segments of length 2^{-k} . A filled square will be composed of 4 sub-squares when halved, 16 when quartered, and $(2^k)^2$ when divided into squares of side-length 2^{-k} . The exponents 1 and 2 in the number of pieces obtained after sub-division are what determine that a line is 1-dimensional and a square is 2-dimensional. For many fractal sets, this exponent will be between 1 and 2, and is what determines one of the ways of defining their dimension.

For modern architecture, understanding dimension goes beyond simply measuring volume, and often involves a concept called *resonance*. In 1940, the Tacoma Narrows Bridge collapsed, not due to high winds, but due to the wind's oscillation being in resonance with the natural oscillation frequency of the structure. The power of resonance was such that it caused the energy to build up, increasing oscillation amplitudes,

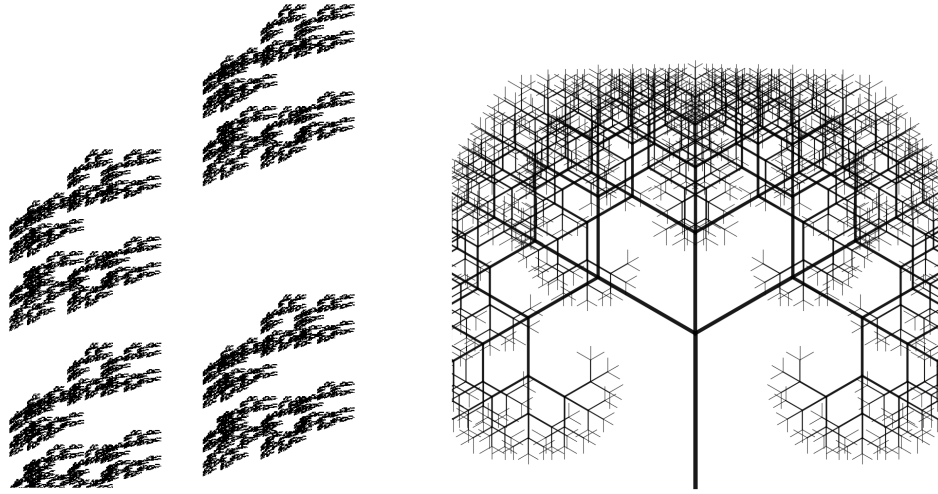


FIGURE 2. Left: A Lalley-Gatzouras carpet C of dimension $\dim_{\text{B}} C \approx 1.65\dots$. Right: A fractal tree T of dimension $\dim_{\text{H}} T \approx 1\dots$.

ending in structural failure. Fourier analysis is the framework for studying these oscillations, tracing its roots back to the early 19th century with Joseph Fourier. While studying the conduction of heat, Fourier proposed that any periodic function could be expressed as a sum of simple sine waves. His controversial and revolutionary work provided a method to break down complex vibrations into fundamental frequencies, using *Fourier coefficients* to quantify how much each sine or cosine contributes to the vibration. By analysing these frequencies and the Fourier coefficients, modern structures are designed so that the resonance of the wind and other natural forces can be *interfered* with to avoid these catastrophic interactions.

While classical dimensions describe how sets fill up space, Fourier analysis allows us to describe how functions are distributed across frequency space, and the decay of their Fourier coefficients is interlinked with notions of smoothness and randomness. By measuring Fourier analytic properties of measures supported on fractal sets, we gain an understanding of how smooth, arithmetically structured, or random the sets or their measures are.

1.1.1. Notation.

Throughout this Thesis we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. If we wish to emphasise that the constant C depends on some parameter λ we shall express it as $A \lesssim_{\lambda} B$ or $A \approx_{\lambda} B$. If $A \subseteq \mathbb{R}^d$ is finite, we write $|A|$ for its cardinality, or its diameter if it is not finite. It will be clear from context whether this notation is referring to cardinality or diameter. For $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm. We denote the d -dimensional Lebesgue measure

with $\mathcal{L}^d$. Finally, for $p, q \in [1, \infty]$ we write p', q' to refer to their Hölder conjugate exponents i.e. $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Chapter-specific notation will be introduced at the beginning of each Chapter.

1.2. Iterated Function Systems

One of the simplest ways of constructing fractal sets is by iterating the application contracting maps via an *iterated function system* (IFS). Repeating this process ad infinitum will give the infinite detail we wish fractal sets to exhibit.

Let $B \subseteq \mathbb{R}^d$ be a compact set. A map $\varphi : B \rightarrow B$ is a *contraction* if there exists $0 < c < 1$ such that

$$(1.2.1) \quad |\varphi(x) - \varphi(y)| \leq c|x - y|,$$

for all $x, y \in B$. A contraction is a *similarity* if (1.2.1) holds with equality, i.e. if there exists $0 < c < 1$ such that $|\varphi(x) - \varphi(y)| = c|x - y|$.

An IFS on $\mathbb{R}^d$ is a finite collection of contractions $\varphi = (\varphi_i)_{i=1}^m$, $m \geq 2$, defined on a common B . We call the unique non-empty compact set $K \subseteq B$ such that

$$K = \bigcup_{i=1}^m \varphi_i(K)$$

the *attractor* of the IFS [57]. When all the maps in φ are similarities, we call the attractor K a *self-similar set*, and if the linear parts of the maps φ_i are all equal, we call the IFS *homogeneous*.

Note that there are no constraints on the contracting maps that define an IFS, these can interact in many ways. Thus, it is often useful to ask for conditions that ensure that the components of the IFS do not overlap much, referred to as separation conditions. We say that an IFS with attractor K satisfies the *strong separation condition* if $\varphi_i(K) \cap \varphi_j(K) = \emptyset$ whenever $i \neq j$, and it satisfies the *open set condition* if there exists a non-empty open set U such that $\bigcup_{i=1}^m \varphi_i(U) \subseteq U$ with the union being disjoint.

Given a probability vector $(p_i)_{i=1}^m$, the measure determined by an IFS φ and $(p_i)_{i=1}^m$ is the unique Borel probability measure μ on K satisfying

$$(1.2.2) \quad \mu = \sum_{i=1}^m p_i \varphi_i \mu.$$

If K is a self-similar set, then μ is called a *self-similar measure*. In this case and with $p_i = \text{Lip}(\varphi_i)^s$, where s is such that $\sum_{i=1}^m \text{Lip}(\varphi_i)^s = 1$ and $\text{Lip}(f)$ denotes the Lipschitz constant of a function f , we refer to μ as the *natural* self-similar measure on the attractor. For a more thorough presentation on iterated function systems and their dimensions we refer the reader to [30, 31].

1.3. Dimensions of Sets

The concept of dimension comes from trying to differentiate between sets in $\mathbb{R}^d$ which have Lebesgue measure zero. Lebesgue measure is not suitable for studying objects at lower dimensions, e.g. curves in $\mathbb{R}^2$ have Lebesgue measure zero, but we still want to measure their lengths. Clearly, if a curve is very wiggly, the diameter will not be a good measure of its length. For this reason, we might consider covering the curve with multiple balls and then adding their diameters together. We are interested in doing this for arbitrarily small balls and studying how the set gets covered as the diameters get smaller.

Given a Borel set $X \subseteq \mathbb{R}^d$ and $\delta > 0$, a δ -cover of X is a countable (or finite) collection of sets $\{U_i\}_i$ with $|U_i| \leq \delta$ such that $X \subseteq \bigcup_i U_i$. Given $s \geq 0$ and $\delta > 0$, define

$$\mathcal{H}_\delta^s(X) = \inf \left\{ \sum_i |U_i|^s : \{U_i\}_i \text{ is a } \delta\text{-cover of } X \right\}.$$

As $\delta \rightarrow 0$ the allowable covers of X are reduced, and so the infimum increases, converging to the number $\mathcal{H}^s(X) \in [0, \infty]$, which we call the s -dimensional Hausdorff measure of X [100]. The mapping $s \mapsto \mathcal{H}^s(X)$ is non-increasing, and there is a critical value $s \in [0, d]$ such that $\mathcal{H}^t(X) = \infty$ for all $t < s$, and $\mathcal{H}^t(X) = 0$ for all $t > s$. This critical value s is called the Hausdorff dimension of X , defined in 1918 by Felix Hausdorff [53],

$$\begin{aligned} \dim_H X &= \sup \left\{ s \in [0, d] : \mathcal{H}^s(X) = \infty \right\} \\ &= \inf \left\{ s \in [0, d] : \mathcal{H}^s(X) = 0 \right\}, \end{aligned}$$

or equivalently (see e.g. [31, Chapter 2])

$$\dim_H X = \inf \left\{ s \in [0, d] : \forall \varepsilon > 0 \exists \text{ a countable cover } \{U_i\}_i \text{ of } X : \sum_i |U_i|^s \leq \varepsilon \right\}.$$

In the definition of the Hausdorff dimension there is some flexibility; the diameters of the sets involved in the covers are allowed to vary ($|U_i|$ need not be equal to $|U_j|$ for $i \neq j$). If we were to force sets in the cover to be of the same size, we would be studying the set at coarser scales. The box dimension does precisely this. Given $X \subseteq \mathbb{R}^d$, its lower box dimension is

$$\begin{aligned} \underline{\dim}_B X &= \inf \left\{ s \in [0, d] : \forall \varepsilon > 0 \exists \text{ a finite cover } \{U_i\}_i \text{ of } X : |U_i| = |U_j| \forall i, j \right. \\ &\quad \left. \text{and } \sum_i |U_i|^s \leq \varepsilon \right\} \\ &= \liminf_{\delta \searrow 0} \frac{\log N_\delta(X)}{-\log \delta}, \end{aligned}$$

where $N_\delta(X)$ is the smallest number of sets of diameter δ needed to cover X . If instead of $\liminf$ above we take the $\limsup$, then we get the *upper box dimension* $\overline{\dim}_B X$. Whenever $\underline{\dim}_B X = \overline{\dim}_B X$ we refer to them as the *box dimension* which is denoted by $\dim_B X$.

In addition, we use the packing dimension of a Borel set $X \subset \mathbb{R}^d$, defined by Claude Tricot [115] in terms of the packing outer measure, or equivalently (see e.g. [31, Theorem 3.8]) as

$$\dim_P X = \inf \left\{ \sup_{i \in \mathbb{N}} \overline{\dim}_B U_i : \{U_i\}_i \text{ countable cover of } X \right\}.$$

The packing dimension plays an important role in the dimension theory of fractal sets, and is dual to the Hausdorff dimension in the sense that many of their properties are interwoven. All the dimensions $\dim$ defined above satisfy that for any α -Hölder map f and $X \subseteq \mathbb{R}^d$, $\dim f(X) \leq \alpha^{-1} \dim X$.

A well-known result in fractal geometry states that if $X \subseteq \mathbb{R}^d$ is the self-similar attractor of an IFS satisfying the open set condition with $0 < \text{Lip}(\varphi_i) < 1$ for each $i = 1, \dots, m$, then $\dim_H X = \dim_B X = \dim_P X = s$, where s is such that

$$\sum_{i=1}^m \text{Lip}(\varphi_i)^s = 1,$$

often known as the Hutchinson-Moran formula (see e.g. [31, 57]).

For any compact $X \subseteq \mathbb{R}^d$, the notions of dimension defined above satisfy

$$(1.3.1) \quad \begin{array}{ccc} & \dim_P X & \\ \swarrow & & \searrow \\ \dim_H X & & \overline{\dim}_B X \\ \nwarrow & & \nearrow \\ & \underline{\dim}_B X & \end{array}$$

1.3.1. Frostman's Lemma and the Potential Theoretic Method.

When considering general fractal sets (not necessarily arising from an IFS), it is often challenging obtaining lower bounds for their Hausdorff dimension. Frostman's lemma makes this task easier (see e.g. [81, Chapter 8]).

THEOREM 1.3.1 (Frostman's Lemma). *Let $0 \leq s \leq d$. For a Borel set $X \subseteq \mathbb{R}^d$, $\mathcal{H}^s(X) > 0$ if and only if there exists a finite Borel measure supported on X such that*

$$(1.3.2) \quad \mu(B(x, r)) \lesssim r^s$$

for all $x \in \mathbb{R}^d$ and $r > 0$.

We say that a measure μ satisfies a *Frostman condition* or that it is *s-Frostman* for some $0 \leq s \leq d$, if it satisfies (1.3.2).

By the definition of the Hausdorff dimension, given $X \subseteq \mathbb{R}^d$, Frostman's lemma implies

$$\begin{aligned} \{s : \mathcal{H}^s(X) = \infty\} &\subset \{s : \mathcal{H}^s(X) > 0\} \\ &= \{s : \exists \mu \text{ finite Borel measure on } X : \mu(B(x, r)) \lesssim r^s\}. \end{aligned}$$

Then, $\dim_{\text{H}} X \leq \sup\{s : \exists \mu \text{ finite Borel measure on } X : \mu(B(x, r)) \lesssim r^s\}$. Also, given that $\mathcal{H}^s(X) > 0$ implies $s \leq \dim_{\text{H}} X$ we get the reverse inequality, thus proving

$$\dim_{\text{H}} X = \sup\{s : \exists \mu \text{ finite Borel measure on } X : \mu(B(x, r)) \lesssim r^s\}.$$

An alternative characterisation of the Hausdorff dimension is given by the potential theoretic method.

Given $0 \leq s \leq d$ the *s-energy* of a Borel measure μ is defined as

$$I_s(\mu) = \iint |x - y|^{-s} d\mu(y) d\mu(x).$$

This energy integral is deeply related to Frostman's lemma. If $\mu(B(x, r)) \leq r^s$, then $I_t(\mu) < \infty$ for any $t < s$. To see this, first note that by

$$\int |x - y|^{-t} d\mu(y) = t \int_0^{\text{diam}(X)} \mu(B(x, r)) r^{-(t+1)} dr.$$

Thus, for any $t < s$,

$$\begin{aligned} I_t(\mu) &= t \int \int_0^{\text{diam}(X)} r^{-(t+1)} \mu(B(x, r)) dr d\mu(x) \\ &\leq t\mu(\mathbb{R}^d) \int_0^{\text{diam}(X)} r^{s-t-1} dr \end{aligned}$$

and the last integral is finite since μ is a finite measure and $t < s$. Note however, that finiteness of $I_s(\mu)$ does not imply $\mu(B(x, r)) \lesssim r^s$. We now show that said implication does hold for a suitable restriction of μ . If $I_s(\mu) < \infty$, then

$$\int |x - y|^{-s} d\mu(y) < \infty$$

for μ -a.e. $y \in \mathbb{R}^d$. Therefore, there exists $0 < M < \infty$ such that the set

$$X_M := \left\{x : \int |x - y|^{-s} d\mu(y) \leq M\right\},$$

has positive μ measure. In effect,

$$\sum_{M=1}^{\infty} \mu(X_M) \geq \mu\left(\bigcup_{M=1}^{\infty} X_M\right) = \mu\left(\left\{x : \int |x-y|^{-s} d\mu(y) = \infty\right\}^c\right) > 0.$$

Let $\nu = \mu|_{X_M}$, $x \in X_M$, and $r > 0$, then

$$(1.3.3) \quad r^{-s} \nu(B(x, r)) = \int_{B(x, r) \cap X_M} r^{-s} d\mu(y) \leq \int_{B(x, r) \cap X_M} |x-y|^{-s} d\mu(y) \leq M.$$

Let $x \in \mathbb{R}^d \setminus X_M$ and $r > 0$. If $B(x, r) \cap X_M = \emptyset$, then $\nu(B(x, r)) = 0$. Otherwise, let $y \in B(x, r) \cap X_M$, using (1.3.3) with $B(x, r) \subseteq B(y, 2r)$,

$$\nu(B(x, r)) \leq \nu(B(y, 2r)) \leq M(2r)^s.$$

This implies that for every $x \in \mathbb{R}^d$, $r > 0$, $\nu(B(x, r)) \lesssim r^s$. With this, we arrive at an alternative definition of the Hausdorff dimension of a Borel set $X \subseteq \mathbb{R}^d$,

$$\dim_{\text{H}} X = \sup\{s \in [0, d] : \exists \mu \text{ Borel finite on } X : I_s(\mu) < \infty\}.$$

1.4. Dimensions of Measures

Although in the previous section we showed that the conditions

$$\mu(B(x, r)) \lesssim r^s \quad \text{and} \quad I_s(\mu) < \infty$$

are related and give rise to equivalent definitions of Hausdorff dimension of a set, for a fixed measure, the two conditions themselves are not equivalent. In fact, these are two of many ways of capturing how a given measure distributes the mass across its support.

We now use these, and other similar conditions, to define notions of fractal dimension for measures. Given a Borel measure μ , its L^∞ -dimension (also called *Frostman dimension*) is given by

$$D_\infty(\mu) = \sup\{\alpha \in [0, d] : \mu(B(x, r)) \lesssim r^\alpha, \forall r > 0, x \in \mathbb{R}^d\}.$$

Likewise, the L^2 -dimension of μ (also called *correlation dimension* or *energy dimension*) is

$$D_2(\mu) = \sup\{s \in [0, d] : I_s(\mu) < \infty\},$$

which quantifies how a measure distributes mass on average.

A natural generalisation of the dimensions defined above are the L^q -dimensions, which quantify fluctuations in the distribution of mass on the set. These dimensions are of particular interest for measures that exhibit multifractal behaviour (see Figure 3), and are known for many particular examples. For $q > 1$, the L^q -dimension of a Borel

measure μ is

$$D_q(\mu) = \sup \left\{ s \in [0, d] : \int \left(\int \frac{d\mu(y)}{|x-y|^s} \right)^{q-1} d\mu(x) < \infty \right\}.$$

There are many characterisations of these dimensions (see e.g. [56, 92, 97]). What is more, in Proposition 5.3.1 we provide another equivalent definition for $q > 1$ that, to the best of our knowledge, is not found elsewhere in the literature. The L^q -dimensions were first introduced in the following ‘lower’ form. For $q \geq 0$ and $r > 0$ let

$$C_q(\mu, r) = \int \mu(B(x, r))^{q-1} d\mu(x).$$

Then for $q \neq 1$ the L^q -dimension of μ is

$$D_q(\mu) = \liminf_{r \rightarrow 0} \frac{\log C_q(\mu, r)}{(q-1) \log r},$$

and for $q = 1$,

$$D_1(\mu) = \liminf_{r \rightarrow 0} \frac{\int \log \mu(B(x, r)) d\mu(x)}{\log r}.$$

The map $q \mapsto D_q(\mu)$ is non-increasing for $q \geq 0$, and continuous except perhaps at $q = 1$. The two definitions given above coincide when $q > 1$ and can be further generalised for $q < 1$. In this Thesis, we shall only concern ourselves with the $q \geq 0$ case. The L^q -dimensions encode useful geometric information about μ , including containing several well-known notions of dimension as special cases. For example:

$$D_0(\mu) = \underline{\dim}_{\text{B}} \text{spt}(\mu); \quad \lim_{q \searrow 1} D_q(\mu) \leq \dim_{\text{H}} \mu; \quad \text{and} \quad \lim_{q \rightarrow \infty} D_q(\mu) = D_{\infty}(\mu),$$

where $\dim_{\text{H}} \mu$ is the (lower) Hausdorff dimension of a Borel measure μ

$$\dim_{\text{H}} \mu = \inf \{ \dim_{\text{H}} X : \mu(X) > 0 \}.$$

Whenever $D_q(\mu)$ is continuous at 1, $\dim_{\text{H}} \text{spt}(\mu) = D_1(\mu)$, otherwise $\lim_{q \nearrow 1} D_q(\mu) \leq \dim_{\text{H}} \mu \leq \lim_{q \searrow 1} D_q(\mu)$. It also always holds that $\dim_{\text{H}} \text{spt}(\mu) \geq \dim_{\text{H}} \mu \geq \lim_{q \nearrow 1} D_q(\mu)$.

In particular, for self-similar measures (1.2.2) on homogeneous IFSs composed of linear similarities $\varphi = \{\lambda x + t_i\}_{i=1}^m$ on $\mathbb{R}$ satisfying the open set condition, the above formulae can be simplified to

$$(1.4.1) \quad D_q(\mu) = \frac{\log \sum_{i=1}^m p_i^q}{(q-1) \log \lambda}$$

for $q \geq 0$, $q \neq 1$, and

$$D_1(\mu) = \frac{\sum_{i=1}^m p_i \log p_i}{\log \lambda}.$$

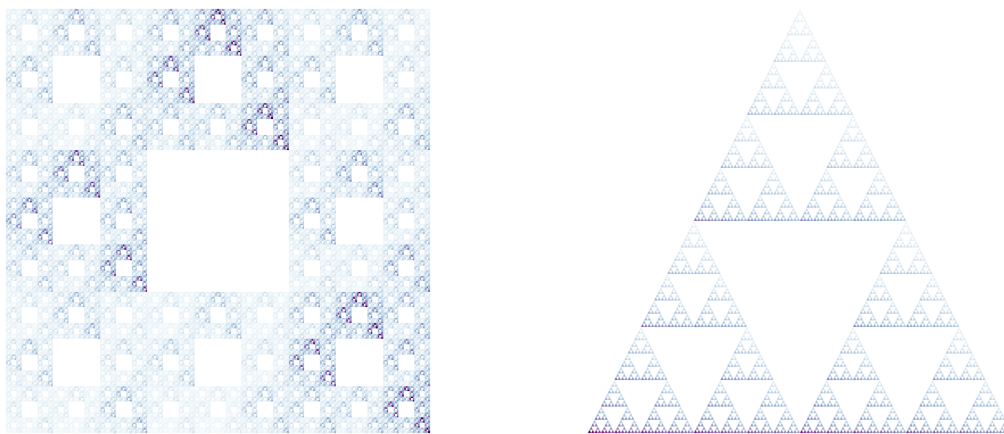


FIGURE 3. Multifractal measures supported on the Sierpiński carpet (left) and the Sierpiński triangle (right). The dark regions represent high concentration of mass and the light low concentration of mass.

1.5. Fourier Analysis and its Role in Fractal Geometry

The Fourier transform is the most important tool in harmonic analysis and comes up naturally in connection with energy integrals and the Hausdorff dimension. Fourier analysis is used throughout this Thesis, in particular, to study this connection further by considering more general energy integrals (see Section 1.5.2), the applications of the Fourier transform to problems in fractal geometry (see Chapter 2), and to study the role that fractal measures play in problems from harmonic analysis (Chapters 3–6).

We begin by introducing the Schwartz class on $\mathbb{R}^d$, denoted by $\mathcal{S}(\mathbb{R}^d)$. We say that a function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is rapidly decreasing if for every $N \in \mathbb{N}$,

$$|f(x)| \lesssim_N |x|^{-N}.$$

The *Schwartz class* is the space of infinitely differentiable functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ such that f and all of its partial derivatives are rapidly decreasing. We will say that f is a *Schwartz function* if $f \in \mathcal{S}(\mathbb{R}^d)$. Note that all functions in $\mathcal{S}(\mathbb{R}^d)$ are also in $L^1(\mathbb{R}^d)$.

The reason for defining the Schwartz class is due to the desirable property of being closed under the Fourier transform. We formally define the *Fourier transform* of a function f as

$$(1.5.1) \quad \widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx.$$

It is easy to see that the integral above converges absolutely provided $f \in \mathcal{S}(\mathbb{R}^d)$ or $f \in L^1(\mathbb{R}^d)$, since $|e^{\pm 2\pi i \xi \cdot x}| = 1$, and this gives a pointwise definition of the Fourier transform of such functions f . Moreover, the Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ onto

itself, and the Fourier inversion theorem states that for any $f \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \widehat{f}(\xi) d\xi = f(x),$$

giving rise to the definition of the *inverse Fourier transform*

$$\check{f}(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} f(\xi) d\xi.$$

There is no general agreement on whether the exponential in either $\widehat{f}$ or $\check{f}$ should have the minus sign, or if the 2π should be in the exponential at all. Sometimes the Fourier transform is defined with a multiplicative constant $1/(2\pi)^d$, other times both the Fourier transform and its inverse are defined with the constant $1/(2\pi)^{d/2}$. Although all these definitions give rise to different operators, the properties they satisfy are the same after performing a change of variables. Throughout this Thesis, we use the definition of Fourier transform as above, with 2π in the exponential to avoid any multiplicative constants.

An important consequence of the Fourier inversion that is also a key property of the Fourier transform is *Parseval's formula*, which states that for any $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$(1.5.2) \quad \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx = \int_{\mathbb{R}^d} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi.$$

In particular, setting $f = g$ in (1.5.2) gives the well-known *Plancherel's theorem* which implies that the Fourier transform is an L^2 isometry on $\mathcal{S}(\mathbb{R}^d)$

$$\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

Since $\mathcal{S}(\mathbb{R}^d)$ is dense on $L^2(\mathbb{R}^d)$, Plancherel's theorem allows us to naturally extend the definition of the Fourier transform to $L^2(\mathbb{R}^d)$.

The space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^d)$ is the space of continuous linear functionals from $\mathcal{S}(\mathbb{R}^d)$ into $\mathbb{C}$. For any $p \in [1, \infty]$, a function $f \in L^p(\mathbb{R}^d)$ can be identified with a functional $T_f \in \mathcal{S}'(\mathbb{R}^d)$ defined as

$$T_f(\varphi) = \int_{\mathbb{R}^d} \varphi(x) f(x) dx.$$

With this, the definition of the Fourier transform can be extended to functions in $L^p(\mathbb{R}^d)$ for $p > 2$ and to functions satisfying $|f(x)| \lesssim |x|^s$ for some $s > 0$, by considering them as tempered distributions. Given a tempered distribution T , its Fourier transform $\widehat{T}$ is the linear map

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. From now on we will abuse notation and write $\widehat{f}$ for the Fourier transform of $f \in L^p$ for any $p \in [1, \infty]$. Note that whenever $p > 2$, both $\widehat{f}$ and

any L^p norm should be thought of in the distributional sense. For a more detailed introduction to the Fourier transform and its basic properties we refer the reader to [48, Chapter 2].

Since (1.5.1) makes sense for any $f \in L^1(\mathbb{R}^d)$, it also does for finite measures. Writing $\mathcal{M}(X)$ for the space of finite, compactly supported, Borel measures on $X \subseteq \mathbb{R}^d$, $L^1(X) \subseteq \mathcal{M}(X)$ via the identification $f \mapsto \mu$, where the measure μ is defined by the formula

$$\mu(A) = \int_A f(x) d\mu(x).$$

Therefore, we can generalise the definition of Fourier transform to include measures $\mu \in \mathcal{M}(\mathbb{R}^d)$ defining it as

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d\mu(x).$$

For any $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\widehat{\mu}$ is uniformly continuous. Indeed, there is a close relation between localisation properties of μ and smoothness of $\widehat{\mu}$ and vice versa. This relation is known as the *Fourier uncertainty principle* (see e.g. [26, 24]).

1.5.1. Dimensions Captured by the Fourier Transform.

One of the most important properties in Fourier analysis is the relation between the Fourier transform and L^2 norms. This is now of particular importance to us given the form of the s -energies previously introduced. Using basic properties of the Fourier transform (see e.g. [48]) for any compactly supported Borel measure on $\mathbb{R}^d$, $0 < s < d$, and letting $\kappa_s(x) = |x|^{-s}$,

$$(1.5.3) \quad I_s(\mu) = \int (\mu * \kappa_s)(x) d\mu(x) = \int_{\mathbb{R}^d} \widehat{\mu * \kappa_s}(\xi) \overline{\widehat{\mu}}(\xi) d\xi = \gamma_{s,d} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi,$$

where $\gamma_{s,d} = \pi^{s-\frac{d}{2}} \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})}$. Some care is required in this computation since $\widehat{\kappa_s}(\xi) = \gamma_{s,d} \kappa_{d-s}(\xi)$ only holds in a distributional sense, and the use of the convolution formula and Parseval's relation need to be properly justified. This justification is omitted from this Thesis and the interested reader is referred to [83, Chapter 3] for an extensive discussion.

The formula above gives rise to a Fourier analytic definition of the Hausdorff dimension, which gives information regarding the Fourier decay of a measure. For example, if μ has polynomial Fourier decay $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$ for $\xi \in \mathbb{R}^d$, then (1.5.3) implies that $I_t(\mu) < \infty$ for all $t < s$ and since the energies characterise the Hausdorff dimension, $\dim_{\text{H}} \text{spt}(\mu) \geq s$. The s -energy can be interpreted as a Sobolev norm. Thus, having finite energy gives information regarding the smoothness of the measure, for this reason

we define the *Sobolev dimension* of a Borel measure μ by,

$$\dim_S \mu = \sup \left\{ s \geq 0 : \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi < \infty \right\}.$$

This concept goes back to Peres–Schlag [96]; see also [83]. Note when $D_2(\mu) < d$, $\dim_S \mu = D_2(\mu)$ and whenever this happens we will use both notions of dimensions interchangeably. The relation between the Hausdorff dimension of sets and the Fourier transform of measures they support motivates the definition of *Fourier dimension* of a Borel measure μ ,

$$\dim_F \mu = \sup \left\{ s \geq 0 : \sup_{\xi \in \mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^s < \infty \right\},$$

and of a Borel set $X \subseteq \mathbb{R}^d$,

$$\dim_F X = \sup \left\{ \min\{\dim_F \mu, d\} : \mu \text{ finite Borel measure on } X \right\}.$$

For any Borel measure μ , $0 \leq \dim_F \mu \leq \dim_S \mu$ and $\dim_H \mu \geq \min\{d, \dim_S \mu\}$, and for a Borel set $X \subseteq \mathbb{R}^d$, $0 \leq \dim_F X \leq \dim_H X \leq d$ and all of these inequalities can be strict in any combination. Sets for which the Fourier and Hausdorff dimensions coincide are called *Salem sets*. Constructing non-trivial deterministic Salem sets is challenging, but random examples abound. The first construction of a Salem set was a random Cantor set in $\mathbb{R}$ of any dimension between 0 and 1 given by Salem [101]. Both the images of compact sets under (fractional) Brownian motion, and its zero sets are almost surely Salem, see [34, 87]. See also Theorem 6.2.7 for the construction of a randomly built Salem set.

The first explicit Salem found set was the set of *α -well approximable numbers*,

$$E(\alpha) = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{x \in [0, 1] : \|kx\| \leq k^{-(1+\alpha)}\}$$

defined by Jarník in [59] who calculated that its Hausdorff dimension is $2/(2 + \alpha)$ for any $\alpha > 0$, where $\|\cdot\|$ denotes the distance to the nearest integer. Its Fourier dimension was calculated by Kaufman [64]. Some other examples of explicit Salem sets in $\mathbb{R}^d$ for $d \geq 2$ can be found in [43] and [47].

The Fourier dimension is highly sensitive to arithmetic structure, smoothness, and curvature. For example, any measure supported on a hyperplane in $\mathbb{R}^d$ has Fourier dimension zero due to a lack of curvature relative to the ambient space. Also, *any* measure on the middle $(1 - 2a)$ Cantor set for $a \leq 1/3$ $1/a \in \mathbb{Z}$ (this includes the middle third Cantor set, for example) has Fourier dimension zero due to too much arithmetic resonance (see [83, Chapter 8.1]). On the other hand, the surface measure on the sphere in $\mathbb{R}^d$ has Fourier dimension (and Hausdorff dimension) $d - 1$ and random measures often have large Fourier dimension (see for example [8, 17]).

A perhaps striking observation regarding the Fourier dimension is that it depends on the ambient space in a way that the other notions of dimension defined thus far do not. As an example consider the set $X = [0, 1] \times \{0\} \subseteq \mathbb{R}^2$. Any measure μ on X will satisfy that $\widehat{\mu}(\xi_1, \xi_2) = \widehat{\mu}(\xi_1, 0)$ and for a fixed ξ_1 and $\xi_2 \rightarrow \infty$, $\widehat{\mu}$ remains constant. This implies that $\dim_F \mu = \dim_F X = 0$, even though 1-dimensional Lebesgue measure on $[0, 1]$ satisfies that $\widehat{\mathcal{L}^1|_{[0,1]}}(\xi) \lesssim |\xi|^{-1}$. More generally, if $X \subseteq \mathbb{R}^k$ is embedded into $\mathbb{R}^d$ for some $k < d$, then $\dim_F X = 0$ in $\mathbb{R}^d$, and likewise if μ is a measure supported on $\mathbb{R}^k$ and is embedded into $\mathbb{R}^d$ for some $k < d$, then $\dim_F \mu = 0$ in $\mathbb{R}^d$.

Calculating the Fourier dimension of fractal sets or measures is notoriously difficult. Beyond the intrinsic interest itself, this question has seen multiple applications, for example: the Fourier uniqueness problem [76], fractal uncertainty principles [27, 26], Fourier restriction estimates (see Chapter 3), and equidistribution properties of numbers in fractal sets [98], among many others.

It is worth noting that both the Fourier and Sobolev dimensions of measures may, in some cases, surpass the Hausdorff dimension of the ambient space. Consider the 1-dimensional Lebesgue measure restricted to $[0, 1]$, $\mathcal{L}^1|_{[0,1]}$. This measure satisfies $\dim_F \mathcal{L}^1|_{[0,1]} = \dim_S \mathcal{L}^1|_{[0,1]} = 2 > \dim_H[0, 1]$. In particular, if $X \subset \mathbb{R}^d$ has non-empty interior, $\dim_H X = d$, but it supports a measure $d\mu(x) = f(x) dx$ for $f \in \mathcal{S}(X)$, which decays faster than the reciprocal of any polynomial, thus $\dim_F \mu = \infty$.

1.5.2. The Fourier Spectrum.

Note that both the s -energy defined before for $0 < s < d$ (in its Fourier analytic version)

$$(1.5.4) \quad I_s(\mu) \approx_{s,d} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} dx$$

and the ‘energy’ that defines the Fourier dimension

$$(1.5.5) \quad \sup_{\xi \in \mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^s,$$

can be thought of as two ends of a spectrum, where (1.5.4) measures average Fourier decay in an L^2 sense, and (1.5.5) measures average Fourier decay in an L^∞ sense. Taking advantage of this, in [37], the author defined the Fourier spectrum, a family of dimensions lying between the Fourier and the Sobolev dimension of measures, and between the Fourier and Hausdorff dimensions of sets. These dimensions measure the average Fourier decay in an L^p sense, where $p = 2/\theta$ for some $\theta \in [0, 1]$. The Fourier spectrum is part of a framework of taking two related notions of fractal dimensions and studying them via a continuum of dimensions that live in-between. This approach

results in a more nuanced understanding of the individual notions of dimension, see [38] for more on dimension interpolation.

To this end, [37] defined the (s, θ) -energies of a finite, Borel measure on $\mathbb{R}^d$, for $\theta \in (0, 1]$ and $s \geq 0$, as

$$\mathcal{J}_{s,\theta}(\mu) = \left(\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} d\xi \right)^{\theta},$$

and for $\theta = 0$,

$$\mathcal{J}_{s,0}(\mu) = \sup_{\xi \in \mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^s.$$

Then the *Fourier spectrum* of μ at θ is

$$\dim_{\mathbb{F}}^{\theta} \mu = \sup \{s \geq 0 : \mathcal{J}_{s,\theta}(\mu) < \infty\},$$

and for each $\theta \in [0, 1]$, $\dim_{\mathbb{F}} \mu \leq \dim_{\mathbb{F}}^{\theta} \mu \leq \dim_{\mathbb{S}} \mu$, with equality on the left if $\theta = 0$ and equality on the right if $\theta = 1$. As a function of θ , $\dim_{\mathbb{F}}^{\theta} \mu$ is non-decreasing, concave, and is continuous for $\theta \in (0, 1]$, and also continuous at $\theta = 0$ provided μ is compactly supported (see [37, Theorem 1.1 and 1.3]).

The *Fourier spectrum* of a Borel set $X \subseteq \mathbb{R}^d$ is

$$\dim_{\mathbb{F}}^{\theta} X = \sup \{ \min \{ \dim_{\mathbb{F}}^{\theta} \mu, d \} : \mu \text{ finite Borel measure on } X \}.$$

Then, for all $\theta \in [0, 1]$, $\dim_{\mathbb{F}} X \leq \dim_{\mathbb{F}}^{\theta} X \leq \dim_{\mathbb{H}} X$, with equality on the left if $\theta = 0$ and equality on the right if $\theta = 1$. Moreover, $\dim_{\mathbb{F}}^{\theta} X$ is continuous for all $\theta \in [0, 1]$ by [37, Theorem 1.5].

The convolution formula for the Fourier transform gives the straightforward property of the Fourier dimension $\dim_{\mathbb{F}}(\mu^{*k}) = k \dim_{\mathbb{F}} \mu$ for any $k \in \mathbb{N}$. Convolutions of measures are of particular interest when studying properties of sumsets (see e.g. Example 6.2.10). The main limitation of the Fourier spectrum is its difficulty to compute. For this reason, finding examples that illustrate the results we obtain can be challenging. The following is a result from [37] that is used multiple times throughout the Thesis to obtain examples of the Fourier spectrum at $\theta = 1/n$ for measures whose dimension of the convolution is known (or easy to calculate) using tools from e.g. Section 1.4. Although this is not useful whenever information about the Fourier spectrum is needed for large values of $\theta \in [0, 1]$, for the results in this Thesis, we only require knowing the Fourier spectrum for small values of $\theta \in [0, 1]$, see e.g. Corollary 2.5.1 or Theorem 3.2.1.

LEMMA 1.5.1 ([37, Lemma 6.2]). *Let μ be a finite Borel measure on $\mathbb{R}^d$. Then*

$$\dim_{\mathbb{F}}^{\theta}(\mu^{*k}) = k \dim_{\mathbb{F}}^{\theta/k} \mu$$

for all $\theta \in [0, 1]$ and all $k \in \mathbb{N}$. In particular

$$\dim_S(\mu^{*k}) = k \dim_F^{1/k} \mu$$

and, provided $\dim_F^\theta \mu$ is continuous at $\theta = 0$ (e.g. if μ is compactly supported),

$$\dim_F \mu = \lim_{k \rightarrow \infty} \frac{\dim_S(\mu^{*k})}{k}.$$

Beyond the applications mentioned in this Thesis (Chapters 2–5), the Fourier spectrum has been used in the distance set problem [37, Section 7], and to obtain the Fourier dimension of the graph of fractional Brownian motion [73, 74].

1.6. Obtaining Dimensions via Fourier Coefficients

In this Section, we present work from [15]. The definition of Fourier spectrum seen in Section 1.5.2 has the disadvantage of needing to know estimates for the Fourier transform of measures for all frequencies. Although this disadvantage is also seen in the Fourier analytic definition of the Hausdorff dimension, in [52] the authors found an alternative representation of the s -energies that accounts for this by considering only Fourier coefficients.

If μ is a finite Borel measure on $\mathbb{R}^d$ with support contained in $(0, 1)^d$, then for all $s \in (0, d)$,

$$(1.6.1) \quad I_s(\mu) \approx_{d,s} |\widehat{\mu}(0)|^2 + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d}.$$

A similar formula for the (s, θ) -energy exists ([37, Proposition 3.1]), but was only established in the case where θ is the reciprocal of a positive integer and $0 < s < d\theta$. In this case,

$$\mathcal{J}_{s,\theta}(\mu) \approx \left(|\widehat{\mu}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \theta\mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} \right)^\theta.$$

This result has a number of drawbacks. It is unsatisfactory to sum over $\theta\mathbb{Z}^d$ and not $\mathbb{Z}^d$ and to require θ to be the reciprocal of an integer, but it is perhaps even more unsatisfactory to require $s < d\theta$, because, for measures with positive Fourier dimension, it is necessary to study the energies for $s > d\theta$ in order to get a complete picture. In Theorem 1.6.1, stated below, we address all the aforementioned drawbacks in the case where the measure is supported on a closed subset of $(0, 1)^d$.

THEOREM 1.6.1. *Let μ be a finite Borel measure on $\mathbb{R}^d$ with support contained in $[\delta, 1 - \delta]^d$ for some $0 < \delta < 1/2$. Then for all $\theta \in (0, 1]$ and $s > 0$,*

$$(1.6.2) \quad \mathcal{J}_{s,\theta}(\mu) \approx_\delta \left(|\widehat{\mu}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} \right)^\theta.$$

In particular,

$$(1.6.3) \quad \dim_{\mathbb{F}}^{\theta} \mu = \sup \left\{ s \in \mathbb{R} : \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d} < \infty \right\}.$$

The requirement that there should exist such a δ in the above is necessary. To see this, let $d = 1$ and μ be the Lebesgue measure on $[0, 1]$. Then $\widehat{\mu}(\xi) = 0$ for all $\xi \in \mathbb{Z} \setminus \{0\}$ but the Sobolev dimension (and thus Fourier spectrum) is finite; in fact $\dim_{\mathbb{F}}^{\theta} \mu = 2$ for all $\theta \in [0, 1]$. Thus, sampling the Fourier transform along the integers might not ‘see’ the relevant fluctuations.

We note that the analogous result indeed holds in the $\theta = 0$ case, dating back to Kahane [62]. In particular, if μ is a measure supported in $[\delta, 1 - \delta]^d$ for some $0 < \delta < 1/2$ and $t > 0$ is such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-t}$$

for $\xi \in \mathbb{Z}^d$, then the same holds (possibly with a different implicit constant) for $\xi \in \mathbb{R}^d$, see [62, Lemma 1, page 252]. Therefore, the Fourier dimension of a measure can be observed by sampling only along the integers. Again the condition that the measure is supported in $[\delta, 1 - \delta]^d$ is necessary and again this can be seen by the Lebesgue measure example.

One (unavoidable) drawback of Theorem 1.6.1 is that the measure must be supported on a region which is bounded away from the boundary of the unit cube. Theoretically this is not a problem since measures with bounded support can always be scaled to satisfy this requirement. However, for certain examples which come supported on, e.g. the whole unit cube, it can be inconvenient. Nonetheless, an explicit discrete formulation is still possible for these measures, but one is forced to consider different Fourier coefficients, in particular, evaluated along $\alpha\mathbb{Z}^d$ instead of $\mathbb{Z}^d$ for some $\alpha > 0$.

COROLLARY 1.6.2. *Let μ be a finite Borel measure on $\mathbb{R}^d$ with compact support of diameter $R > 0$. Then for all $0 < \alpha < 1/R$ it holds that for all $\theta \in (0, 1]$,*

$$\mathcal{I}_{s,\theta}(\mu) \approx_{\alpha} \left(|\widehat{\mu}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \alpha\mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d} \right)^{\theta}.$$

In particular,

$$\dim_{\mathbb{F}}^{\theta} \mu = \sup \left\{ s \in \mathbb{R} : \sum_{\xi \in \alpha\mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d} < \infty \right\}$$

and also (for completeness)

$$\dim_F \mu = \sup \left\{ s \in \mathbb{R} : \sup_{z \in \alpha \mathbb{Z}^d} |\widehat{\mu}(\xi)|^2 |\xi|^s < \infty \right\}.$$

PROOF. Let $0 < \alpha < 1/R$. Then there exists a translation $\beta \in \mathbb{R}^d$ such that the function ϕ defined as $\phi(\xi) = \alpha\xi + \beta$ satisfies that $\phi(\text{spt}\mu) \subseteq [\delta, 1 - \delta]^d$ for some $0 < \delta < 1/2$ depending only on α . Then, writing $\phi(\mu) = \mu \circ \phi^{-1}$, $|\widehat{\phi(\mu)}(\xi)| \approx_\alpha |\widehat{\mu}(\alpha\xi)|$. Therefore, by a simple change of variables,

$$\mathcal{J}_{s,\theta}(\mu) \approx_\alpha \mathcal{J}_{s,\theta}(\phi(\mu))$$

and, by Theorem 1.6.1,

$$\begin{aligned} \mathcal{J}_{s,\theta}(\phi(\mu))^{1/\theta} &\approx_\alpha |\widehat{\phi(\mu)}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\phi(\mu)}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d} \\ &\approx_\alpha |\widehat{\mu}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\alpha\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d} \\ &\approx_\alpha |\widehat{\mu}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \alpha \mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d}, \end{aligned}$$

which proves the first claim. The formulation of the Fourier spectrum for $\theta \in (0, 1]$ is then immediate. For the case $\theta = 0$, the formulation of the Fourier dimension follows from [62, Lemma 1, page 252]. In particular, if $t > 0$ is such that

$$|\widehat{\phi(\mu)}(\xi)| \approx_\alpha |\widehat{\mu}(\alpha\xi)| \lesssim |\alpha\xi|^{-\frac{t}{2}} \approx_\alpha |\xi|^{-\frac{t}{2}}$$

for $\xi \in \mathbb{Z}^d$, then $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\frac{t}{2}}$ for $\xi \in \mathbb{R}^d$. $\square$

In addition to obtaining a simpler definition for the Fourier spectrum, one of the main motivations behind Theorem 1.6.1 is that it provides us with an easy proof of a general bound for the Fourier spectrum of a Borel measure μ .

THEOREM 1.6.3. *Let μ be a finite Borel measure with compact support on $\mathbb{R}^d$. Then for all $\theta \in [0, 1]$,*

$$(1.6.4) \quad \dim_F^\theta \mu \leq \dim_F \mu + d\theta.$$

PROOF. For $\theta = 0$ or for $\dim_F \mu = \infty$ the result is trivial, so we only consider the other cases. Let $R < \infty$ denote the diameter of the support of μ . Let $\theta \in (0, 1]$ and $\dim_F \mu < t < \infty$. It then follows from Corollary 1.6.2 that, for some $0 < \alpha < 1/R$, there exists a sequence $(\xi_n)_{n \in \mathbb{N}} \subseteq \alpha \mathbb{Z}^d$ with $|\xi_n| \rightarrow \infty$ and such that for all $n \in \mathbb{N}$,

$$|\widehat{\mu}(\xi_n)| \gtrsim |\xi_n|^{-\frac{t}{2}}.$$

Then, using Corollary 1.6.2 again,

$$\mathcal{J}_{s,\theta}(\mu)^{1/\theta} \approx |\widehat{\mu}(0)|^{\frac{2}{\theta}} + \sum_{\xi \in \alpha\mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} \gtrsim \sum_{n \in \mathbb{N}} |\xi_n|^{\frac{-t+s}{\theta}-d} = \infty,$$

whenever $s \geq t + d\theta$. Therefore, $\dim_{\mathbb{F}}^{\theta} \mu \leq \dim_{\mathbb{F}} \mu + d\theta$, as required. $\square$

This new characterisation of the Fourier spectrum of a measure in terms of its Fourier coefficients emphasises the relation to the Fourier ℓ^p dimensions, recently introduced in [120]. These dimensions are defined for each $p > 0$ as

$$(1.6.5) \quad \dim_{\ell^p} \mu = \sup \left\{ s \geq 0 : \sum_{|\xi| \leq R} |\widehat{\mu}(\xi)|^p \lesssim R^{d-s} \right\}.$$

While both the Fourier spectrum and the Fourier ℓ^p dimensions capture smoothness of a measure μ through its Fourier decay, there is a clear distinction between the definitions (1.6.3) and (1.6.5). Since the Fourier ℓ^p dimensions consider a growth rate rather than an infinite sum, (1.6.5) will always satisfy $\dim_{\ell^p} \mu \leq d$, whereas as previously discussed in Section 1.5.1, the Fourier dimension (and thus the Fourier spectrum) can satisfy $\dim_{\mathbb{F}} \mu > d$. The Fourier ℓ^p dimensions are related to the averaged Fourier dimensions $\overline{F}_{\mu}(\theta)$ and $\underline{F}_{\mu}(\theta)$ defined in [37] for $\theta \in (0, 1]$ by

$$\overline{F}_{\mu}(\theta) = \limsup_{R \rightarrow \infty} \frac{\theta \log \left(R^{-d} \int_{|\xi| \leq R} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} d\xi \right)}{-\log R},$$

and

$$\underline{F}_{\mu}(\theta) = \liminf_{R \rightarrow \infty} \frac{\theta \log \left(R^{-d} \int_{|\xi| \leq R} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} d\xi \right)}{-\log R}.$$

In the range $p \geq 1$, $\overline{F}_{\mu}(\theta) = \theta \dim_{\ell^{\frac{2}{\theta}}} \mu$. What is more, for any $\theta \in (0, 1]$, if $\underline{F}_{\mu}(\theta) < d\theta$ then $\dim_{\mathbb{F}}^{\theta} \mu = \underline{F}_{\mu}(\theta)$, and therefore in this case $\overline{F}_{\mu}(\theta) = \theta \dim_{\ell^{\frac{2}{\theta}}} \mu \geq \dim_{\mathbb{F}}^{\theta} \mu$. The applicability of the Fourier ℓ^p dimensions to problems in Diophantine approximation and missing-digit sets (see e.g. [121, 23, 22]) implies that the Fourier spectrum is likely to also be applicable for number-theoretic problems.

1.7. Structure of the Thesis

The structure of this Thesis is as follows. Chapter 2 will be concerned with using the Fourier spectrum to obtain a new estimate for the Hausdorff dimension of the exceptional set of directions for the orthogonal projection, and is based on the work presented in [39]. This estimate has some flexibility that can be leveraged to understand for what sets the known estimates are not optimal.

In Chapter 3, we use the Fourier spectrum to extend the well-known Stein–Tomas theorem, giving a continuum of estimates that can be optimised over. As a result,

we obtain a novel estimate that depends on the Sobolev and Fourier dimensions of the measures. These estimates are applied to several examples, such as measures on curves, the cone, and certain measures on fractal sets, based on [14]. Of particular interest is the application of these estimates to the graph of the Brownian sheet. This is the canonical example of a fractal surface, and it sits in between the fractal and smooth settings, since it has topological co-dimension 1 but still exhibits many fractal properties. This example (for the additive graph) will be the content of Chapter 4 with material from [40].

Next, in Chapter 5 we present unpublished work [16] where we generalised our previous result on Fourier restriction to the L^q -dimensions, which are the natural dimensions used to study measures exhibiting multifractal behaviour. We show that both the Stein–Tomas theorem and our previous result using only the Sobolev and Fourier dimensions are just two endpoints in a spectrum of estimates that depend continuously on the parameter q in the L^q -dimensions. We apply this to a family of random measures exhibiting stochastic self-similarity, the Mandelbrot cascade measure.

As part of an effort to understand the Fourier restriction theory beyond smooth surfaces, in Chapter 6, we present work from [91] on generalising multilinear Fourier extension estimates from smooth to fractal measures. We find several notions of transversality between fractal measures which allow us to determine necessary and sufficient conditions for these estimates to hold.

CHAPTER 2

Orthogonal projections

2.1. Introduction

Orthogonal projections are a central topic in fractal geometry. Marstrand's celebrated projection theorem [79] states that for almost all directions, the orthogonal projection of a Borel set X onto a k -dimensional plane will equal $\min\{k, \dim_{\text{H}} X\}$. However, there may exist many directions for which this does not hold. This naturally leads to the problem of bounding from above the dimension of the set of exceptional directions. This question has attracted a lot of attention since Kaufman's contributions [63]. However, after almost 6 decades and in spite of recent and significant progress, the picture is still not complete. We refer the reader to [32, 82] for a more thorough presentation on the work of Marstrand and its consequences in fractal geometry. A short summary of recent research is given in Section 2.1.2.

2.1.1. Notation.

Given $X \subseteq \mathbb{R}^d$, we shall write $\mathcal{M}(X)$ for the set of Borel measures μ compactly supported on X and such that $0 < \mu(X) < \infty$.

For integers $1 \leq k < d$, we write $G(d, k)$ for the Grassmannian manifold of k -dimensional planes $V \subseteq \mathbb{R}^d$ equipped with the invariant Borel probability measure $\gamma_{d,k}$ obtained from the Haar measure on the topological group of rotations around the origin. We write $P_V : \mathbb{R}^d \rightarrow V$ for the orthogonal projection onto $V \in G(d, k)$, which we identify with $\mathbb{R}^k$. Given a finite, compactly supported Borel measure μ on $\mathbb{R}^d$, we will write μ_V for the push-forward measure under P_V , i.e. $\mu_V(B) = \mu(P_V^{-1}(B))$ for Borel sets $B \subseteq \mathbb{R}^k$. Given $y \in \mathbb{R}^k$ and $V \in G(d, k)$, we define $y_V \in \mathbb{R}^d$ by

$$(2.1.1) \quad \{y_V\} = V \cap P_V^{-1}(y).$$

2.1.2. Hausdorff dimension and orthogonal projections.

One of the most well-known and influential results in Fractal Geometry is Marstrand's projection theorem. This was proved in the plane by Marstrand [79] and in higher dimensions by Mattila [80] and states that for Borel sets $X \subseteq \mathbb{R}^d$ and $\gamma_{d,k}$ almost all $V \in G(d, k)$, $\dim_{\text{H}} P_V(X) = \min\{k, \dim_{\text{H}} X\}$.

Marstrand's projection theorem motivated the study of the dimension of the set of projections for which the conclusion does not hold: the set of exceptional projections. Kaufman [63] addressed this question in the context of compact sets $X \subseteq \mathbb{R}^2$, and proved that for $u \in [0, \dim_{\text{H}} X]$,

$$(2.1.2) \quad \dim_{\text{H}}\{V \in G(2, 1) : \dim_{\text{H}} P_V(X) < u\} \leq u,$$

which was proven to be sharp when $u = \dim_{\text{H}} X$ by Kaufman and Mattila in [65]. Kaufman's proof of (2.1.2) was easy to generalise in (see e.g. [83, Theorem 5.1]) to show that if $X \subseteq \mathbb{R}^d$ has $\dim_{\text{H}} X \leq 1$ then for all $u \in [0, \dim_{\text{H}} X]$,

$$(2.1.3) \quad \dim_{\text{H}}\{V \in G(d, 1) : \dim_{\text{H}} P_V(X) < u\} \leq d - 2 + u.$$

Later, using different methods, Bourgain [10, Theorem 4] and Oberlin [90, Theorem 1.2] proved that for Borel sets $X \subseteq \mathbb{R}^2$,

$$(2.1.4) \quad \dim_{\text{H}}\{V \in G(2, 1) : \dim_{\text{H}} P_V(X) < \tfrac{1}{2} \dim_{\text{H}} X\} = 0,$$

and Oberlin conjectured that if $\frac{\dim_{\text{H}} X}{2} \leq u \leq \min\{\dim_{\text{H}} X, 1\}$, then

$$(2.1.5) \quad \dim_{\text{H}}\{V \in G(2, 1) : \dim_{\text{H}} P_V(X) < u\} \leq 2u - \dim_{\text{H}} X.$$

Oberlin's conjecture (2.1.5) was recently proved in a breakthrough paper of Ren and Wang [99, Theorem 1.2], which built on significant recent progress from various people; see, for example, [49, 93, 94, 109] and the references therein. In fact, [10, Theorem 4] provides a stronger 'ε-improvement' on (2.1.4), and this played an important role in the development of the problem, including in [99].

Most of these results have analogues to higher-dimensional projections onto planes $V \in G(d, k)$ for $k \geq 2$. For example, [80] (see [80, Remarks 5.7 (1)], the proof is written in detail in [83, Corollary 5.12]) generalised (2.1.3) to show that if $\dim_{\text{H}} X \leq k$, then for any $u \in [0, \dim_{\text{H}} X]$

$$(2.1.6) \quad \dim_{\text{H}}\{V \in G(d, k) : \dim_{\text{H}} P_V(X) < u\} \leq k(d - k - 1) + u.$$

This was shown to be sharp when $u = \dim_{\text{H}} X \leq k$ in [65, Theorem 5]. Using capacities, Falconer [29, Theorem 1 (ii)] gave the following upper bound for Borel sets $X \subseteq \mathbb{R}^d$ without restrictions on their Hausdorff dimension, which was also obtained by [96, Proposition 6.1] with a different method. For $u \in [0, k]$,

$$(2.1.7) \quad \dim_{\text{H}}\{V \in G(d, k) : \dim_{\text{H}} P_V(X) < u\} \leq k(d - k) + u - \dim_{\text{H}} X,$$

which was shown to be sharp for $u = k$ in [83, Example 5.13]. Note that Peres and Schlag's bound is only better than (2.1.6) when $\dim_{\text{H}} X > k$. Although their bound is stated only for measures, for the same reason that we give in the proof of Theorem 2.4.1, the result holds for sets.

Bourgain's estimate (2.1.4) was generalised by He in [54], and more recently by [46] who showed that for Borel sets $X \subseteq \mathbb{R}^d$,

$$(2.1.8) \quad \dim_{\mathrm{H}}\{V \in G(d, k) : \dim_{\mathrm{H}} P_V(X) < k \dim_{\mathrm{H}} X/d\} \leq k(d - k) - \min\{k, d - k\}.$$

An important (and simple) observation is that for Salem sets, there are no exceptional directions for the projections, see the beginning of Section 2.2.1. In fact, one can say slightly more. If $X \subseteq \mathbb{R}^d$ is a Borel set with $\dim_{\mathrm{F}} X = t$, then for all $u \leq t$,

$$\{V \in G(d, k) : \dim_{\mathrm{H}} P_V(X) < \min\{k, u\}\} = \emptyset.$$

However, we show in Proposition 2.2.2 that nothing can be said for $u > t$ using the Fourier dimension alone.

This Chapter is structured as follows. We begin by analysing the well-studied case of the exceptional set for the Hausdorff dimension of projections, giving in Lemma 2.2.1 an example that establishes the (well-known folklore) sharpness of the bound proven by Ren and Wang in [99]. After studying the role of the Fourier dimension and its limitations in this setting, in Sections 2.3 and 2.4, we obtain non-trivial bounds for the Fourier spectrum of projections (Theorem 2.3.1 and Proposition 2.3.2), and give a bound for the Hausdorff dimension of the exceptional set of directions for the Fourier spectrum (Theorem 2.4.1).

Despite the estimate from Theorem 2.4.1 not being sharp in general, we show in Section 2.5 that it can be used to improve on sharp exceptional set estimates for the *Hausdorff* dimension in certain cases: if the Fourier analytic information captured by the Fourier spectrum is ‘strong enough’, then non-trivial improvements can be made to many exceptional set estimates, see Proposition 2.3.3 and Corollaries 2.5.1 and 2.5.2. The sharp exceptional set estimate in the plane was recently obtained by Ren–Wang [99] but in higher dimensions less is known. We are able to improve the state-of-the-art in higher dimensions under only very mild assumptions on the set itself, see Corollary 2.5.4, and Section 2.6 where we consider an illustrative example.

2.2. Fourier dimension and exceptional sets for the Hausdorff dimension

We begin by building an example to show the pointwise sharpness of the inequality (2.1.5) proven in [99, Theorem 1.2]. The construction, which was described to us by Tuomas Orponen, is similar to the one given in [65], however, more care is required since there are more parameters to look after.

LEMMA 2.2.1. *For all $s \in (0, 1]$ and all $u \in (0, s)$ there exists a compact set $X_{u,s} \subseteq \mathbb{R}^2$ with $\dim_{\mathrm{H}} X_{u,s} = s$ such that*

$$\dim_{\mathrm{H}}\{V \in G(2, 1) : \dim_{\mathrm{H}} P_V(X_{u,s}) \leq u\} = \max\{0, 2u - s\}.$$

PROOF. Let $s \in (0, 1]$. If $u < s/2$ then by (2.1.4) we know that the dimension of the exceptional set is 0. Thus, we only consider the case $u \in [s/2, s)$. Let $(\eta_m)_{m \in \mathbb{N}}$ be a rapidly increasing sequence of positive integers; say $\eta_{m+1} \geq \eta_m^m$, and define the sets

$$\begin{aligned} A &= \{x \in [0, 1] : \forall m \in \mathbb{N}, d(x, \eta_m^{-u}\mathbb{Z}) \leq \eta_m^{-1}\}; \\ B &= \{x \in [0, 1] : \forall m \in \mathbb{N}, d(x, \eta_m^{-(s-u)}\mathbb{Z}) \leq \eta_m^{-1}\}; \\ C &= \{x \in [0, 1] : \forall m \in \mathbb{N}, d(x, \eta_m^{-(2u-s)}\mathbb{Z}) \leq \eta_m^{-1}\}. \end{aligned}$$

Here $d(x, Y) = \inf\{|x - y| : y \in Y\}$. By [28, Theorem 10] these sets have $\dim_{\text{H}} A = u$, $\dim_{\text{H}} B = s - u$ and $\dim_{\text{H}} C = 2u - s$, and, since their Hausdorff and packing dimensions are equal,

$$\dim_{\text{H}}(A \times B) = \dim_{\text{H}} A + \dim_{\text{H}} B = s.$$

By Marstrand's theorem, since $s \leq 1$, then $\dim_{\text{H}} P_V(A \times B) = s$ for $\gamma_{2,1}$ almost all $V \in G(2, 1)$. Also, $\dim_{\text{H}}(A + cB) = 2u - s$ for all $c \in C$. To prove this, fix $m \in \mathbb{N}$ and define the discrete sets

$$\begin{aligned} A_m &= \{\eta_m^{-u}z : z \in \mathbb{Z}\}; \\ B_m &= \{\eta_m^{-(s-u)}z : z \in \mathbb{Z}\}; \\ C_m &= \{\eta_m^{-(2u-s)}z : z \in \mathbb{Z}\}. \end{aligned}$$

The projection of a point of the form $(\eta_m^{-u}z_1, \eta_m^{-(s-u)}z_2) \in A_m \times B_m$ onto a line with slope $\eta_m^{-(2u-s)}z_3 \in C_m$ is, up to a constant scaling factor, of the form

$$(\eta_m^{-u}z_1, \eta_m^{-(s-u)}z_2) \cdot (1, \eta_m^{-(2u-s)}z_3) = \eta_m^{-u}z_1 + \eta_m^{-u}(z_2z_3) = \eta_m^{-u}(z_1 + z_2z_3).$$

This implies that, for each $m \in \mathbb{N}$, the projections of $A_m \times B_m$ onto lines with slopes given by C_m are contained in A_m .

Now, for each $\eta_m^{-(2u-s)}z_3 \in C_m$, consider the interval of length $2\eta_m^{-1}$ around $\eta_m^{-(2u-s)}z_3$. The union of these intervals constitutes a set of directions for which the projection of $A \times B$ has dimension at most $\dim_{\text{H}} A = u$ at scale η_m^{-1} (that is, the projection may be covered by a constant times η_m^u many intervals of length η_m^{-1}). Intersecting these sets over all $m \in \mathbb{N}$ we get the set C . This shows that for the directions with slopes given by C , the Hausdorff dimension of the projection of $A \times B$ is at most $\dim_{\text{H}} A = u$. It then follows that the exceptional set

$$\{V \in G(2, 1) : \dim_{\text{H}} P_V(A \times B) \leq u\},$$

contains a copy of C . Thus,

$$\dim_{\text{H}}\{V \in G(2, 1) : \dim_{\text{H}} P_V(A \times B) \leq u\} \geq \dim_{\text{H}} C$$

$$\begin{aligned}
&= 2u - s \\
&= 2u - \dim_{\mathbb{H}}(A \times B).
\end{aligned}$$

Since the reverse inequality is true by [99, Theorem 1.2], letting $X_{u,s} = A \times B$ we have the desired result. $\square$

2.2.1. Fourier dimension and orthogonal projections.

Given a finite, compactly supported, Borel measure μ on $\mathbb{R}^d$, an integer $1 \leq k < d$, and $V \in G(d, k)$,

$$(2.2.1) \quad \widehat{\mu_V}(y) = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot P_V(x)} d\mu(x) = \int_{\mathbb{R}^d} e^{-2\pi i y_V \cdot x} d\mu(x) = \widehat{\mu}(y_V)$$

recalling the definition of y_V from (2.1.1). Thus, for all $V \in G(d, k)$ and finite Borel measures, $\dim_{\mathbb{F}} \mu \leq \dim_{\mathbb{F}} \mu_V$ which implies that if $X \subseteq \mathbb{R}^d$ is a Borel set, then for all $V \in G(d, k)$, $\dim_{\mathbb{H}} P_V(X) \geq \dim_{\mathbb{F}} P_V(X) \geq \min\{k, \dim_{\mathbb{F}} X\}$. In particular, for Salem sets, there are no exceptions to Marstrand's theorem. We show in the following proposition that one cannot say anything more than this using the Fourier dimension alone. That is, knowledge of the Fourier dimension of X does not give any information about the dimension of the set of V for which $\dim_{\mathbb{H}} P_V(X) < u$ as soon as $u > \dim_{\mathbb{F}} X$.

PROPOSITION 2.2.2. *For any $s \in (0, 1]$ and $t \in (s/2, s)$ there exists a compact set $X \subseteq \mathbb{R}^2$ with $\dim_{\mathbb{H}} X = s$ and $\dim_{\mathbb{F}} X = t$ such that for $u < t$,*

$$\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(X) \leq u\} = \emptyset,$$

and for $u \geq t$,

$$\dim_{\mathbb{H}}\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(X) \leq u\} \geq 2t - s.$$

That is, the dimension of the exceptional set has a jump discontinuity at $\dim_{\mathbb{F}} X$ from 0 to the largest value it could possibly take according to (2.1.5).

PROOF. Let $s \in (0, 1]$, $t \in (s/2, s)$ and $A := X_{t,s} \subseteq \mathbb{R}^2$ be as in Lemma 2.2.1, that is, A is compact, $\dim_{\mathbb{H}} A = s$, and

$$(2.2.2) \quad \dim_{\mathbb{H}}\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(A) \leq t\} = 2t - s.$$

Using [43], define a compact Salem set $B \subseteq \mathbb{R}^2$ of dimension t and assume that A and B are disjoint. Clearly $\dim_{\mathbb{F}} A \leq t$ (otherwise (2.2.2) could not hold) and so $\dim_{\mathbb{F}}(A \cup B) = t$, and for each $u < t$,

$$\dim_{\mathbb{H}}\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(A \cup B) \leq u\} = 0.$$

Also,

$$\dim_{\mathbb{H}} P_V(A \cup B) = \max\{\dim_{\mathbb{H}} P_V(A), \dim_{\mathbb{H}} P_V(B)\},$$

and by the discussion at the beginning of Section 2.2.1, $\dim_{\mathbb{H}} P_V(B) = t$ for all $V \in G(2, 1)$. Then,

$$\dim_{\mathbb{H}}\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(A \cup B) \leq t\} = \dim_{\mathbb{H}}\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(A) \leq t\} = 2t - s$$

and therefore, if $u \geq t$,

$$\dim_{\mathbb{H}}\{V \in G(2, 1) : \dim_{\mathbb{H}} P_V(A \cup B) \leq u\} \geq 2t - s.$$

Letting $X = A \cup B$ gives the result. $\square$

This last proposition showed us that the Fourier dimension alone is perhaps not strong enough to capture the effect that the decay of the Fourier transform of measures has on projections. We will see later that the Fourier spectrum can say more, see Proposition 2.3.3, Theorem 2.4.1 and Corollary 2.5.1.

2.3. Projection theorems for the Fourier spectrum

Marstrand type theorems for other dimensions are well studied. For example, there are Marstrand type theorems for the box and packing dimensions, where the dimension of the projection is almost surely constant. For a more detailed discussion of such results we refer the reader to [32, 82] and the references therein. On the other hand, in the case of the Assouad dimension, a surprising result holds: the Assouad dimension of orthogonal projections need not be almost surely constant [41].

It is natural to try to answer these types of questions for the Fourier spectrum (see Section 1.5.2). A useful lower bound can be obtained for almost all directions following Kaufman's potential-theoretic proof of Marstrand's theorem, [63].

THEOREM 2.3.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $X \subseteq \mathbb{R}^d$ be a Borel set and $1 \leq k < d$ be an integer. Then, for $\gamma_{d,k}$ almost all $V \in G(d, k)$, for all $\theta \in (0, 1]$, $\dim_{\mathbb{F}}^{\theta} \mu_V \geq \dim_{\mathbb{F}}^{\theta} \mu$ and $\dim_{\mathbb{F}}^{\theta} P_V(X) \geq \min\{k, \dim_{\mathbb{F}}^{\theta} X\}$.*

PROOF. First fix $\theta \in (0, 1]$. Integrating the energies of the projected measure with respect to planes in $G(d, k)$, recalling (2.1.1) and (2.2.1), and applying Fubini's theorem, gives

$$\begin{aligned} \int_{G(d,k)} \mathcal{J}_{s,\theta}(\mu_V)^{1/\theta} d\gamma_{d,k}(V) &= \int_{G(d,k)} \int_{\mathbb{R}^k} |\widehat{\mu_V}(y)|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} dy d\gamma_{d,k}(V) \\ &= \int_{\mathbb{R}^k} \int_{G(d,k)} |\widehat{\mu}(y_V)|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\gamma_{d,k}(V) dy. \end{aligned}$$

We now continue by decomposing Lebesgue measure on $\mathbb{R}^d$ in terms of Lebesgue measure on $\mathbb{R}^{k-1}$ and a measure on $\mathbb{R}S^{d-k}$. For ease of exposition, we present the underlying method behind the Blaschke-Petkantschin formula.

Note that $\mathbb{R}^d = \mathbb{R}^{d-k+1} \times \mathbb{R}^{k-1} = (\mathbb{R}S^{d-k}) \times \mathbb{R}^{k-1}$, where S^{d-k} denotes the sphere in $\mathbb{R}^{d-k+1}$ and $\mathbb{R}^{d-k+1} = \mathbb{R}S^{d-k} = \{re : r \in \mathbb{R}, e \in S^{d-k}\}$ is the usual representation of $\mathbb{R}^{d-k+1}$ in spherical coordinates. Fix $y \in \mathbb{R}^k$ and let $\pi_y : G(d, k) \rightarrow S^{d-k}$ be defined by $\pi_y(V) = e$ where $e \in S^{d-k}$ is chosen such that $y_V = y_{W_e}$ for $W_e := \{(re, v) : r \in \mathbb{R}, v \in \mathbb{R}^{k-1}\} \in G(d, k)$. Note that, for almost all $y \in \mathbb{R}^k$, the choice of e is unique for all $V \in G(d, k)$ and so π_y is a well-defined Borel measurable function almost surely. By $O(d-k)$ rotational symmetry, the push-forward of $\gamma_{d,k}$ by π_y (for almost all y) is simply σ^{d-k} ; the surface measure on the sphere S^{d-k} . Therefore, for almost all $y \in \mathbb{R}^k$, by the disintegration theorem for measures, for each $e \in S^{d-k}$ there exists a σ^{d-k} almost everywhere uniquely determined family of Borel probability measures $\{\gamma_{y,e}\}_{e \in S^{d-k}}$ on $G(d, k)$ such that

$$\int_{G(d,k)} |\widehat{\mu}(y_V)|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\gamma_{d,k}(V) = \int_{S^{d-k}} \int_{\pi_y^{-1}(e)} |\widehat{\mu}(y_V)|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\gamma_{y,e}(V) d\sigma^{d-k}(e).$$

Therefore, since $y_V = y_{W_e}$ only depends on e ,

$$\int_{G(d,k)} |\widehat{\mu}(y_V)|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\gamma_{d,k}(V) = \int_{S^{d-k}} |\widehat{\mu}(y_{W_e})|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\sigma^{d-k}(e).$$

Thus, disintegrating Lebesgue measure on $\mathbb{R}^d = (\mathbb{R}S^{d-k}) \times \mathbb{R}^{k-1} = \cup_{e \in S^{d-k}} W_e$ using σ^{d-k} on S^{d-k} and $\mathcal{H}^k$ on W_e as $d\xi \approx |y|^{d-k} d\mathcal{H}^k(y) d\sigma^{d-k}(e)$, we get

$$\begin{aligned} \int_{G(d,k)} \mathcal{J}_{s,\theta}(\mu_V)^{1/\theta} d\gamma_{d,k}(V) &= \int_{\mathbb{R}^k} \int_{S^{d-k}} |\widehat{\mu}(y_{W_e})|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\sigma^{d-k}(e) dy \\ &\approx \int_{S^{d-k}} \int_{W_e} |\widehat{\mu}(y)|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-k} d\mathcal{H}^k(y) d\sigma^{d-k}(e) \\ &\approx \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} d\xi \\ &= \mathcal{J}_{s,\theta}(\mu)^{1/\theta}. \end{aligned}$$

Then, if $s < \dim_{\mathbb{F}}^{\theta} \mu$, $\mathcal{J}_{s,\theta}(\mu) < \infty$ and for $\gamma_{d,k}$ almost all $V \in G(d, k)$, $\mathcal{J}_{s,\theta}(\mu_V) < \infty$. Therefore, $\dim_{\mathbb{F}}^{\theta} \mu_V \geq s$ for $\gamma_{d,k}$ almost all $V \in G(d, k)$. This proves the statement for measures holds *pointwise*, that is, for all $\theta \in (0, 1]$ it holds for $\gamma_{d,k}$ almost all $V \in G(d, k)$. However, since the Fourier spectrum of a compactly supported measure is continuous, and thus determined on a countable set, we can immediately upgrade this pointwise result such that for $\gamma_{d,k}$ almost all $V \in G(d, k)$ it holds *simultaneously* for all θ , as required.

The claim for sets follows immediately from the claim for measures. $\square$

Recall that when $\theta = 0$ the result of the previous theorem holds for *all* $V \in G(d, k)$, and when $\theta = 1$ the reverse inequality for sets is valid for *all* $V \in G(d, k)$ due to the fact that projections are Lipschitz maps.

The proof of Theorem 2.3.1 is rather simpler in the case $k = 1$ since then $G(d, k)$ may be identified with S^{d-1} and $\mathbb{R}^d$ thus expressed in spherical coordinates as $\mathbb{R}^d = \mathbb{R}G(d, 1)$. The complication for $k \geq 2$ arises because parametrising $\mathbb{R}^d$ by points in $G(d, k) \times \mathbb{R}^k$ leads to ‘redundancy’ which must then be integrated out. Setting $\theta = 1$ and recalling the simple fact that the Hausdorff dimension cannot increase under orthogonal projections yields Marstrand’s projection theorem for sets. To the best of our knowledge this is a new proof of Marstrand’s theorem in the case $k \geq 2$; see [83, proof of Theorem 4.1] for the $\theta = k = 1$ case, treated in the same (but simpler) way.

From Theorem 1.6.3 we know that $\dim_F^\theta \mu \leq \dim_F \mu + d\theta$. Combining this with the previous argument gives for all $V \in G(d, k)$ and $\theta \in [0, 1]$,

$$\dim_F^\theta \mu \leq \dim_F \mu + d\theta \leq \dim_F \mu_V + d\theta \leq \dim_F^\theta \mu_V + d\theta.$$

The following proposition shows that in fact, a stronger inequality holds.

PROPOSITION 2.3.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $X \subseteq \mathbb{R}^d$ be a Borel set, $\theta \in [0, 1]$ and $1 \leq k < d$ an integer. For all $V \in G(d, k)$, $\dim_F^\theta \mu_V \geq \dim_F^\theta \mu - (d - k)\theta$ and $\dim_F^\theta P_V(X) \geq \min\{k, \dim_F^\theta X - (d - k)\theta\}$.*

PROOF. If $\theta = 0$ the result is trivial by the discussion at the beginning of Section 2.2.1. Let R be larger than the diameter of the support of both μ and μ_V and $0 < \alpha < 1/R$. By Corollary 1.6.2, we have for any $s \geq 0$ and $\theta \in (0, 1]$,

$$\mathcal{J}_{s,\theta}(\mu)^{1/\theta} \approx 1 + \sum_{\xi \in \alpha\mathbb{Z}^d \setminus \{0\}} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta} - d}.$$

Therefore, identifying $V \in G(d, k)$ with $\mathbb{R}^k$, applying Corollary 1.6.2 again, and restricting the summation to $\alpha\mathbb{Z}^k \setminus \{0\}$,

$$\mathcal{J}_{s,\theta}(\mu)^{1/\theta} \gtrsim 1 + \sum_{\xi \in \alpha\mathbb{Z}^k \setminus \{0\}} |\widehat{\mu_V}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s - (d-k)\theta}{\theta} - k} \approx \mathcal{J}_{s-(d-k)\theta,\theta}(\mu_V)^{1/\theta},$$

which proves the result.

The claim for sets follows immediately from the claim for measures. $\square$

As an immediate consequence of the previous proposition, we get some new non-trivial information about the exceptional set for Hausdorff dimension.

PROPOSITION 2.3.3. *Let $X \subseteq \mathbb{R}^d$ be Borel and $1 \leq k < d$ an integer. If $u \leq \sup_{\theta \in [0,1]} (\dim_F^\theta X - (d - k)\theta)$, then*

$$\{V \in G(d, k) : \dim_H P_V(X) < u\} = \emptyset.$$

PROOF. Since $\dim_F^\theta P_V(X) \leq \dim_H P_V(X)$ for all $\theta \in [0, 1]$, then

$$\{V \in G(d, k) : \dim_H P_V(X) < u\} \subseteq \{V \in G(d, k) : \dim_F^\theta P_V(X) < u\}.$$

Thus, the result follows from Proposition 2.3.2. $\square$

With this, if $X \subseteq \mathbb{R}^2$ is a Borel set such that for some $\theta \in [0, 1]$, $\theta < \dim_F^\theta X$, we get an improvement on the Bourgain–Oberlin bound (2.1.4) in the sense that the exceptional set is empty, not just of Hausdorff dimension 0, once u is small enough. Furthermore, if for some $\theta \in [0, 1]$, $\frac{\dim_H X}{2} + \theta < \dim_F^\theta X$, then we get an improved range in the Bourgain–Oberlin bound (in addition to the upgrade from dimension 0 to empty). Given that for Borel sets in $\mathbb{R}^2$, $\dim_F^\theta X \leq \min\{\dim_F X + 2\theta, \dim_H X\}$ by Theorem 1.6.3, this latter improvement is only possible for $\frac{\dim_H X}{2} - \dim_F X < \theta < \frac{\dim_H X}{2}$. In particular, we must have $\dim_F X > 0$ but, if $\dim_F X < \frac{\dim_H X}{2}$, then the improvement will not come from the Fourier dimension directly and can only be achieved by the Fourier spectrum.

QUESTION 2.3.4. *Is there a Marstrand theorem for the Fourier spectrum? More precisely, fix a Borel set or finite Borel measure in $\mathbb{R}^d$, an integer $1 \leq k < d$ and $\theta \in [0, 1]$: is it true that the value of the Fourier spectrum at θ of the projection onto V is the same for almost all $V \in G(d, k)$?*

An intriguing special case of this question concerns the Fourier dimension (that is, when $\theta = 0$) and we are unaware of any progress on this front. On the other hand, for $\theta = 1$ we know the answer to the question is yes for sets by Marstrand’s theorem and for finite, compactly supported, Borel measures μ on $\mathbb{R}^d$ with $\dim_S \mu \leq k$ by [56, Theorem 1.1]. We note that if this ‘pointwise’ question could be answered in the affirmative then, using continuity of the Fourier spectrum, the result could be upgraded to hold almost surely, for all $\theta \in [0, 1]$ simultaneously.

2.4. Exceptional set estimates for the Fourier spectrum

Recall Proposition 2.2.2, in which we showed that the Fourier dimension does not help improve the bound of the exceptional set of projections for values $u > \dim_F X$. The following theorem shows that the Fourier spectrum can do better. When $\theta = 1$ we recover the bound from [96, Proposition 6.1], which also inspired our proof.

THEOREM 2.4.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\theta \in (0, 1]$ and $1 \leq k < d$ be an integer. Then for all $0 \leq u \leq \min\{k, \dim_F^\theta \mu\}$,*

$$(2.4.1) \quad \dim_H\{V \in G(d, k) : \dim_F^\theta \mu_V < u\} \leq \max\left\{0, k(d - k) + \frac{u - \dim_F^\theta \mu}{\theta}\right\}.$$

Furthermore, if X is a Borel set in $\mathbb{R}^d$ and $\theta \in (0, 1]$, then for all $0 \leq u \leq \min\{k, \dim_F^\theta X\}$,

$$\dim_H\{V \in G(d, k) : \dim_F^\theta P_V(X) < u\} \leq \max \left\{ 0, k(d-k) + \frac{u - \dim_F^\theta X}{\theta} \right\}$$

PROOF. The claim for sets follows from the statement for measures. To see this fix $\theta \in (0, 1]$, let $\varepsilon > 0$ and $\mu \in \mathcal{M}(X)$ be such that $\dim_F^\theta \mu \geq \dim_F^\theta X - \varepsilon$. Since $\dim_F^\theta P_V(X) \geq \min\{k, \dim_F^\theta \mu_V\}$ and $u \leq k$, it follows that $\dim_F^\theta P_V(X) < u \Rightarrow \dim_F^\theta \mu_V < u$. Then,

$$\begin{aligned} \dim_H\{V \in G(d, k) : \dim_F^\theta P_V(X) < u\} &\leq \dim_H\{V \in G(d, k) : \dim_F^\theta \mu_V < u\} \\ &\leq k(d-k) + \frac{u - \dim_F^\theta \mu}{\theta} \\ &= k(d-k) + \frac{u - \dim_F^\theta X + \varepsilon}{\theta}, \end{aligned}$$

and letting $\varepsilon \rightarrow 0$ gives the result. We now proceed to prove the claim for measures, which follows the general strategy of [96]; see also [83].

Let $G_{u,\theta} = \{V \in G(d, k) : \dim_F^\theta \mu_V < u\}$ and suppose (2.4.1) is false for some $u > 0$. Choose $\tau > 0$ such that $k(d-k) + \frac{u-s}{\theta} < \tau < \dim_H G_{u,\theta}$, for some $s < \dim_F^\theta \mu$. First, observe that $G_{u,\theta}$ is a Borel set. Indeed,

$$\begin{aligned} G(d, k) \setminus G_{u,\theta} &= \bigcap_{\varepsilon \in (0,1) \cap \mathbb{Q}} \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \left\{ V \in G(d, k) : \int_{|y| < m} |\widehat{\mu_V}(y)|^{\frac{2}{\theta}} |y|^{\frac{u-\varepsilon}{\theta}-k} dy < n \right\} \\ &=: \bigcap_{\varepsilon \in (0,1) \cap \mathbb{Q}} \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} A_{\varepsilon,n,m} \end{aligned}$$

and since, recalling (2.2.1), $V \mapsto \widehat{\mu_V}(y) = \widehat{\mu}(y_V)$ is continuous (since the Fourier transform of μ is continuous and y_V depends continuously on V) the set $A_{\varepsilon,n,m}$ is open and thus $G_{u,\theta}$ is Borel. Therefore, by Frostman's lemma (Theorem 1.3.1) there exists a measure $\nu \in \mathcal{M}(G_{u,\theta})$ such that $\nu(B(V, r)) \leq r^\tau$ for all $V \in G(d, k)$ and $r > 0$. We will show that

$$(2.4.2) \quad \int_{G(d,k)} \mathcal{J}_{u,\theta}(\mu_V)^{1/\theta} d\nu(V) < \infty,$$

and this will imply that $\mathcal{J}_{u,\theta}(\mu_V)^{1/\theta} < \infty$ for ν almost every $V \in G(d, k)$. Then $\nu(G_{u,\theta}) = 0$ which contradicts the fact that $\nu \in \mathcal{M}(G_{u,\theta})$.

We write $\mathcal{S}(\mathbb{R}^d)$ for the family of functions in the Schwartz class on $\mathbb{R}^d$, see [83, Chapter 3]. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, such that $\varphi(x) = 1$ in $\text{spt}\mu$, where $\text{spt}\mu$ denotes the (compact) support of μ . Then $\mu = \varphi\mu$ and $\widehat{\mu} = \widehat{\varphi\mu} = \widehat{\varphi} * \widehat{\mu}$. Moreover, $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$, and for every $N \in \mathbb{N}$, $|\widehat{\varphi}(\xi)| \lesssim_{\varphi,N} (1+|\xi|)^{-N}$. Using Hölder's inequality with conjugate

exponents $2/\theta$ and $2/(2-\theta)$, we obtain the following estimate for $\xi \in \mathbb{R}^d$.

$$\begin{aligned}
|\widehat{\mu}(\xi)|^{\frac{2}{\theta}} &\leq \left[\int_{\mathbb{R}^d} |\widehat{\mu}(\xi - y) \widehat{\varphi}(y)| dy \right]^{\frac{2}{\theta}} \\
&= \left[\int_{\mathbb{R}^d} |\widehat{\mu}(\xi - y)| |\widehat{\varphi}(y)|^{\frac{\theta}{2}} |\widehat{\varphi}(y)|^{\frac{2-\theta}{2}} dy \right]^{\frac{2}{\theta}} \\
&\leq \left[\left(\int_{\mathbb{R}^d} |\widehat{\mu}(\xi - y)|^{\frac{2}{\theta}} |\widehat{\varphi}(y)| dy \right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^d} |\widehat{\varphi}(y)| dy \right)^{\frac{2-\theta}{2}} \right]^{\frac{2}{\theta}} \\
&= \int_{\mathbb{R}^d} |\widehat{\mu}(\xi - y)|^{\frac{2}{\theta}} |\widehat{\varphi}(y)| dy \left(\int_{\mathbb{R}^d} |\widehat{\varphi}(y)| dy \right)^{\frac{2-\theta}{\theta}} \\
&\lesssim \int_{\mathbb{R}^d} |\widehat{\mu}(\xi - y)|^{\frac{2}{\theta}} |\widehat{\varphi}(y)| dy = (|\widehat{\varphi}| * |\widehat{\mu}|^{\frac{2}{\theta}})(\xi).
\end{aligned}$$

By (2.2.1), $\widehat{\mu}_V(y) = \widehat{\mu}(y_V)$, and so we have the following estimate:

$$\begin{aligned}
&\int_{G(d,k)} \mathcal{J}_{u,\theta}(\mu_V)^{1/\theta} d\nu(V) \\
&= \int_{G(d,k)} \int_{\mathbb{R}^k} |\widehat{\mu}_V(y)|^{\frac{2}{\theta}} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&= \int_{G(d,k)} \int_{\mathbb{R}^k} |\widehat{\mu}(y_V)|^{\frac{2}{\theta}} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&\lesssim \int_{G(d,k)} \int_{\mathbb{R}^k} (|\widehat{\varphi}| * |\widehat{\mu}|^{\frac{2}{\theta}})(y_V) |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&= \int_{G(d,k)} \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^d} |\widehat{\varphi}(y_V - \xi)| |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} d\xi \right) |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&= \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} \left[\int_{G(d,k)} \int_{\mathbb{R}^k} |\widehat{\varphi}(y_V - \xi)| |y|^{\frac{u}{\theta}-k} dy d\nu(V) \right] d\xi \\
&\lesssim \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} \left[\int_{G(d,k)} \int_{\mathbb{R}^k} (1 + |y_V - \xi|)^{-N} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \right] d\xi.
\end{aligned}$$

To finish the proof of the theorem we need to show that the last integral is finite. It is enough to see that for N sufficiently large,

$$(2.4.3) \quad \int_{G(d,k)} \int_{\mathbb{R}^k} (1 + |y_V - \xi|)^{-N} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \lesssim |\xi|^{\frac{u}{\theta}+k(d-k)-d-\tau},$$

for all $\xi \in \mathbb{R}^d$ with $|\xi| \geq 2$ because then, since $k(d-k) + \frac{u-s}{\theta} < \tau$,

$$\begin{aligned}
\int_{G(d,k)} \mathcal{J}_{u,\theta}(\mu_V)^{1/\theta} d\nu(V) &\lesssim \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{u}{\theta}+k(d-k)-d-\tau} d\xi \\
&\lesssim \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} d\xi
\end{aligned}$$

$$= \mathcal{J}_{s,\theta}(\mu)^{1/\theta} < \infty,$$

since $s < \dim_{\mathbb{F}}^{\theta} \mu$. This establishes (2.4.2) and completes the proof.

To prove (2.4.3) note that from the definition of ν we have for all $r > 0$ and $\xi \in \mathbb{R}^d$ (see [83, (5.11)]),

$$(2.4.4) \quad \nu(\{V \in G(d, k) : d(\xi, V) \leq r\}) \lesssim \left(\frac{r}{|\xi|}\right)^{\tau-(k-1)(d-k)}$$

where here and in what follows, $d(\xi, Y) = \inf\{|\xi - y| : y \in Y\}$. Fix $\xi \in \mathbb{R}^d$ with $|\xi| \geq 2$ and split the integral into the dyadic annuli centred at ξ as follows:

$$\begin{aligned} \int_{G(d,k)} \int_{\mathbb{R}^k} (1 + |y_V - \xi|)^{-N} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\ = \iint_{\{(V,y): |y_V - \xi| \leq 1/2\}} + \sum_{\{j \geq 0: |\xi| > 2^{j+1}\}} \iint_{\{(V,y): 2^{j-1} < |y_V - \xi| \leq 2^j\}} \\ + \sum_{\{j \geq 0: |\xi| \leq 2^{j+1}\}} \iint_{\{(V,y): 2^{j-1} < |y_V - \xi| \leq 2^j\}} \end{aligned}$$

where the sums are over integer j . For clarity, let us analyse each of the terms separately.

First term: In this case we have $|y| \approx |\xi|$, the inclusion

$$\{(V, y) : |y_V - \xi| \leq 1/2\} \subseteq \{(V, y) : d(\xi, V) \leq 1/2\},$$

and the trivial estimate

$$\int_{\mathbb{R}^k} (1 + |y_V - \xi|)^{-N} dy \lesssim 1$$

which holds for N sufficiently large. These three things yield

$$\begin{aligned} \iint_{\{(V,y): |y_V - \xi| \leq 1/2\}} (1 + |y_V - \xi|)^{-N} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\ \lesssim |\xi|^{\frac{u}{\theta}-k} \int_{\{V: d(\xi, V) \leq 1/2\}} \int_{\mathbb{R}^k} (1 + |y_V - \xi|)^{-N} dy d\nu(V) \\ \lesssim |\xi|^{\frac{u}{\theta}-k} \nu(\{V : d(\xi, V) \leq 1/2\}) \\ \lesssim |\xi|^{\frac{u}{\theta}-k-\tau+(k-1)(d-k)} \\ (2.4.5) \quad = |\xi|^{\frac{u}{\theta}+k(d-k)-d-\tau} \end{aligned}$$

by (2.4.4).

Second term: Since $|y_V - \xi| \leq |\xi|/2$, we again have $|y| \approx |\xi|$. Then

$$\sum_{\{j \geq 0: |\xi| > 2^{j+1}\}} \iint_{\{(V,y): 2^{j-1} < |y_V - \xi| \leq 2^j\}} (1 + |y_V - \xi|)^{-N} |y|^{\frac{u}{\theta}-k} dy d\nu(V)$$

$$\begin{aligned}
&\lesssim |\xi|^{\frac{u}{\theta}-k} \sum_{\{j \geq 0: |\xi| > 2^{j+1}\}} 2^{-jN} \iint_{\{(V,y): 2^{j-1} < |y_V - \xi| \leq 2^j\}} dy d\nu(V) \\
&\lesssim |\xi|^{\frac{u}{\theta}-k} \sum_{\{j \geq 0: |\xi| > 2^{j+1}\}} 2^{-jN} 2^{kj} \nu(\{V : d(\xi, V) \leq 2^j\}) \\
&\lesssim |\xi|^{\frac{u}{\theta}-k-\tau+(k-1)(d-k)} \sum_{j=0}^{\infty} 2^{j(k+\tau-(k-1)(d-k)-N)} \quad (\text{by (2.4.4)}) \\
(2.4.6) \quad &\lesssim |\xi|^{\frac{u}{\theta}+k(d-k)-d-\tau}
\end{aligned}$$

provided $N > k + \tau - (k-1)(d-k) = d + \tau - k(d-k)$.

Third term: Using that, for each relevant j , $|y| \leq 2^j + |\xi| \leq 3 \cdot 2^j$ and $\nu(G(d, k)) \lesssim 1$,

$$\begin{aligned}
&\sum_{\{j \geq 0: |\xi| \leq 2^{j+1}\}} \iint_{\{(V,y): 2^{j-1} < |y_V - \xi| \leq 2^j\}} (1 + |y_V - \xi|)^{-N} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&\lesssim \sum_{\{j \geq 0: |\xi| \leq 2^{j+1}\}} 2^{-jN} \iint_{\{(V,y): 2^{j-1} < |y_V - \xi| \leq 2^j\}} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&\lesssim \sum_{\{j \geq 0: |\xi| \leq 2^{j+1}\}} 2^{-jN} \int_{G(d,k)} \int_{\{y: |y| \leq 3 \cdot 2^j\}} |y|^{\frac{u}{\theta}-k} dy d\nu(V) \\
&\lesssim \sum_{\{j \geq 0: |\xi| \leq 2^{j+1}\}} 2^{-jN} \int_{\{y: |y| \leq 3 \cdot 2^j\}} |y|^{\frac{u}{\theta}-k} dy \\
&\lesssim \sum_{\{j \geq 0: |\xi| \leq 2^{j+1}\}} 2^j \left(\frac{u}{\theta} - N\right) \\
&\lesssim |\xi|^{\frac{u}{\theta}-N} \\
(2.4.7) \quad &\leq |\xi|^{\frac{u}{\theta}+k(d-k)-d-\tau}
\end{aligned}$$

provided $N > \max\{d + \tau - k(d-k), u/\theta\}$. Combining (2.4.5), (2.4.6) and (2.4.7) we have the theorem. $\square$

2.5. Applications

2.5.1. Bounding the exceptional set for Hausdorff dimension.

Let $X \subseteq \mathbb{R}^d$ be a Borel set and $\mu \in \mathcal{M}(\mathbb{R}^d)$. Since for all $\theta \in [0, 1]$, $\min\{d, \dim_{\mathbb{F}}^{\theta} \mu\} \leq \dim_{\mathbb{H}} \mu$ and $\dim_{\mathbb{F}}^{\theta} X \leq \dim_{\mathbb{H}} X$, we can use Theorem 2.4.1 to bound the exceptional set for the *Hausdorff* dimension. This corollary can be viewed as our main result, even though Theorem 2.4.1 is stronger.

COROLLARY 2.5.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $1 \leq k < d$ be an integer. Then for all $u \in [0, k]$,*

$$\dim_{\mathrm{H}}\{V \in G(d, k) : \dim_{\mathrm{H}} \mu_V < u\} \leq \max \left\{ 0, k(d - k) + \inf_{\theta \in (0, 1]} \frac{u - \dim_{\mathrm{F}}^{\theta} \mu}{\theta} \right\}.$$

Furthermore, if X is a Borel set in $\mathbb{R}^d$, then for all $u \in [0, k]$,

$$\dim_{\mathrm{H}}\{V \in G(d, k) : \dim_{\mathrm{H}} P_V(X) < u\} \leq \max \left\{ 0, k(d - k) + \inf_{\theta \in (0, 1]} \frac{u - \dim_{\mathrm{F}}^{\theta} X}{\theta} \right\}.$$

Writing the n -fold convolution as μ^{*n} for $n \in \mathbb{N}$, we obtain the following corollary as a direct application of Lemma 1.5.1. This formulates our main result above for measures in terms of the Sobolev dimension of convolutions.

COROLLARY 2.5.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $1 \leq k < d$ be an integer. Then for all $u \in [0, k]$,*

$$\dim_{\mathrm{H}}\{V \in G(d, k) : \dim_{\mathrm{H}} \mu_V < u\} \leq \max \left\{ 0, k(d - k) + \inf_{n \in \mathbb{N}} (nu - \dim_{\mathrm{S}}(\mu^{*n})) \right\}.$$

This corollary will be useful in Section 2.6, where we calculate the Fourier spectrum of an explicit self-affine measure at $\theta = 1/2$ and use it to bound the exceptional set of projections for the Hausdorff dimension of a self-affine set. That example shows that the Fourier spectrum can be used to improve on the state-of-the-art bounds in higher dimensions in a representative situation, which we discuss in more detail in the next subsections.

2.5.1.1. Improving the sharp bounds in dimension 2. With Corollary 2.5.1 in mind, and specialising only to the case of Borel sets $X \subseteq \mathbb{R}^2$, we ask for conditions under which we are able to improve Ren–Wang’s sharp inequality [99, Theorem 1.2]; recall (2.1.5). Indeed, this will happen for $X \subseteq \mathbb{R}^2$ provided

$$(2.5.1) \quad 1 + \frac{u - \dim_{\mathrm{F}}^{\theta} X}{\theta} < 2u - \dim_{\mathrm{H}} X$$

for some $\theta \in (0, 1]$ and some u in the range $\frac{\dim_{\mathrm{H}} X}{2} < u \leq \min\{\dim_{\mathrm{H}} X, 1\}$.

First consider the case when $\dim_{\mathrm{H}} X < 1$, see Figure 4: Left. Since $\dim_{\mathrm{F}}^{\theta} X \leq \dim_{\mathrm{H}} X$, (2.5.1) is only possible for $\theta < \frac{1}{2}$ in which case we require

$$\dim_{\mathrm{F}}^{\theta} X > \frac{\dim_{\mathrm{H}} X}{2} + \theta.$$

This would then give improvements on (2.1.5) for u satisfying

$$\frac{\dim_{\mathrm{H}} X}{2} < u < \min \left\{ \frac{\dim_{\mathrm{F}}^{\theta} X - \theta(1 + \dim_{\mathrm{H}} X)}{1 - 2\theta}, \dim_{\mathrm{H}} X \right\}.$$

Next consider the case when $\dim_{\text{H}} X \geq 1$, see Figure 4: Right. For $\theta < \frac{1}{2}$, we (again) require

$$(2.5.2) \quad \dim_{\text{F}}^{\theta} X > \frac{\dim_{\text{H}} X}{2} + \theta$$

and this gives improvements on (2.1.5) for u satisfying

$$\frac{\dim_{\text{H}} X}{2} < u < \min \left\{ \frac{\dim_{\text{F}}^{\theta} X - \theta(1 + \dim_{\text{H}} X)}{1 - 2\theta}, 1 \right\}.$$

On the other hand, for $\theta > \frac{1}{2}$ we require

$$\dim_{\text{F}}^{\theta} X > 1 + \theta(\dim_{\text{H}} X - 1)$$

and this gives improvements on (2.1.5) for u satisfying

$$\max \left\{ \frac{\theta(1 + \dim_{\text{H}} X) - \dim_{\text{F}}^{\theta} X}{2\theta - 1}, \frac{\dim_{\text{H}} X}{2} \right\} < u < 1.$$

Curiously, for $\theta = \frac{1}{2}$ we get improvement for all u as long as

$$(2.5.3) \quad \dim_{\text{F}}^{1/2} X > \frac{1 + \dim_{\text{H}} X}{2}.$$

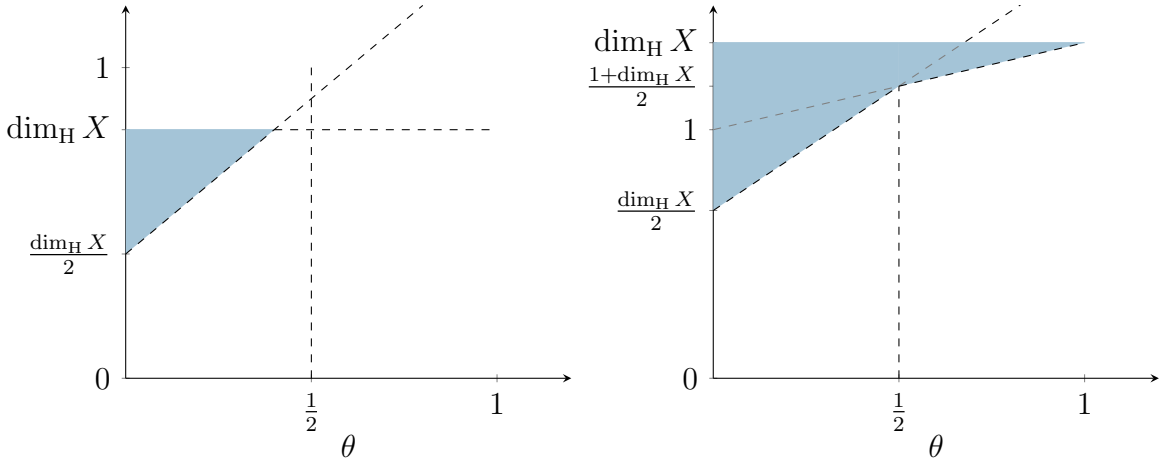


FIGURE 4. In order to improve Ren–Wang’s exceptional set estimate for a Borel set $X \subseteq \mathbb{R}^2$, we need the Fourier spectrum of X to intersect the shaded region, i.e. for some $\theta \in [0, 1]$ the point $(\theta, \dim_{\text{F}}^{\theta} X)$ lies in the shaded region. Left: when $\dim_{\text{H}} X < 1$. Right: when $\dim_{\text{H}} X \geq 1$.

It is easily seen in Figure 4 that if $\dim_{\text{F}} X > \dim_{\text{H}} X/2$, then we get improvement on (2.1.5), but we already knew this from the discussion in Section 2.2.1. However, what Figure 4 now reveals is that this is not needed and, in fact, improvement can be gained for sets with small Fourier dimension but large Fourier spectrum. Important to observe here is that the slope of the lines bounding the shaded region is never more than

1, whereas the slope of the Fourier spectrum of a subset of the plane can be as large as 2, see Theorem 1.6.3 (that is, the Fourier spectrum can ‘catch up’ even if the Fourier dimension is too small to give improvement by itself). It is more or less clear that examples admitting improvement of (2.1.5) via the Fourier spectrum abound. That said, building explicit examples is not completely straightforward, partly because it is difficult to compute the Fourier spectrum in specific settings.

Another interesting feature of Figure 4 is that if $\dim_{\mathbb{H}} X \leq 1$, then the Fourier spectrum can only be used to improve (2.1.5) if $\theta < \frac{1}{2} \dim_{\mathbb{H}} X \leq 1/2$ but if $\dim_{\mathbb{H}} X \geq 1$, then the whole range of θ is potentially relevant. One reason for this could be that in the case $\dim_{\mathbb{H}} X < 1$, the bound given in Theorem 2.4.1 for $\theta = 1$ is not sharp, as was mentioned in Section 2.1.2. This is one of the reasons why it would be interesting to obtain a Fourier spectrum analogue of the sharp bounds.

By scrutinising the limit as $\theta \rightarrow 1$ we get a useful corollary.

COROLLARY 2.5.3. *Let $X \subseteq \mathbb{R}^2$ be a Borel set with $\dim_{\mathbb{H}} X > 1$ and*

$$D = \overline{\partial_-} \dim_{\mathbb{F}}^{\theta} X \big|_{\theta=1} = \limsup_{\theta \rightarrow 1} \frac{\dim_{\mathbb{H}} X - \dim_{\mathbb{F}}^{\theta} X}{1 - \theta}$$

be the upper left semi-derivative of $\dim_{\mathbb{F}}^{\theta} X$ at $\theta = 1$. If $D < \dim_{\mathbb{H}} X - 1$, then there is necessarily improvement on Ren–Wang’s inequality (2.1.5).

2.5.1.2. Improving the known bounds in higher dimensions. We now consider the higher dimensional case, where the sharp bounds are not known. Let $X \subseteq \mathbb{R}^d$, for $d \geq 3$, be a Borel set and $1 \leq k < d$ be an integer. We can use Corollary 2.5.1 to improve Mattila’s (2.1.6) and Falconer’s (2.1.7) when $\dim_{\mathbb{H}} X \leq k$ and $\dim_{\mathbb{H}} X > k$, respectively, and Gan’s (2.1.8) when $u = \frac{k}{d} \dim_{\mathbb{H}} X$.

Let us first consider the case $\dim_{\mathbb{H}} X \leq k$; see Figure 5: Left. Then we want to have for some $\theta \in (0, 1]$ and some $0 < u \leq \dim_{\mathbb{H}} X$,

$$k(d - k) + \frac{u - \dim_{\mathbb{F}}^{\theta} X}{\theta} < k(d - k - 1) + u,$$

which yields

$$u(1 - \theta) + k\theta < \dim_{\mathbb{F}}^{\theta} X.$$

Note that no improvement will be possible when $u = \dim_{\mathbb{H}} X$, that is, when (2.1.6) is sharp. However, for smaller values of u (especially for u close to zero) improvement is possible, and even quite common; see Corollaries 2.5.4 and Lemma 2.5.5.

Similarly, in the case $\dim_{\mathbb{H}} X \geq k$ we can improve Falconer’s bound (2.1.7); see Figure 5: Right. This would happen if

$$(2.5.4) \quad k(d - k) + \frac{u - \dim_{\mathbb{F}}^{\theta} X}{\theta} < k(d - k) + u - \dim_{\mathbb{H}} X,$$

for some $\theta \in (0, 1]$ and some u in the range $0 \leq u \leq k$. This gives

$$(2.5.5) \quad u(1 - \theta) + \theta \dim_{\mathbb{H}} X < \dim_{\mathbb{F}}^{\theta} X.$$

This time improvement is possible for any relevant value of u . In fact, it turns out that the Fourier spectrum will lead to an improvement over the known bounds for ‘most’ sets in some sense. This is captured by the following corollary.

COROLLARY 2.5.4. *Let $X \subseteq \mathbb{R}^d$ be a Borel set. If $\dim_{\mathbb{F}}^{\theta} X > \theta \max\{k, \dim_{\mathbb{H}} X\}$ for some $\theta \in [0, 1)$ then there is necessarily improvement on Falconer’s inequality (2.1.7) and Mattila’s inequality (2.1.6) for all sufficiently small $u > 0$.*

Important to note in the above is that it *always* holds that $\dim_{\mathbb{F}}^{\theta} X \geq \theta \dim_{\mathbb{H}} X$; see [37, Theorem 1.1]. We now show that one only has to impose very mild conditions on X in order to get $\dim_{\mathbb{F}}^{\theta} X > \theta \max\{k, \dim_{\mathbb{H}} X\}$ for all sufficiently small θ . First of all, note that we get strict inequality close to $\theta = 0$ immediately if the Fourier dimension is strictly positive, and so we focus on the case where the Fourier dimension is zero. We use recent work of Khalil [67] to get good information about the Fourier spectrum.

Following [67], for $C \geq 1$ and $\alpha > 0$, a Borel measure on $\mathbb{R}^d$ is (C, α) -uniformly affinely non-concentrated if for every $\varepsilon > 0$, $x \in \mathbb{R}^d$, $0 < r \leq 1$ and hyperplane $W \subseteq \mathbb{R}^d$

$$(2.5.6) \quad \mu(W^{(\varepsilon r)} \cap B(x, r)) \leq C\varepsilon^{\alpha} \mu(B(x, r)),$$

where $W^{(r)}$ is the r -neighbourhood of W . This condition will be satisfied very generally. For example, when $d = 1$ asking that $X \subseteq \mathbb{R}$ supports such a measure is simply asking for X to be uniformly perfect. In higher dimensions the condition is satisfied widely for IFS attractors not lying in a hyperplane (including self-similar sets, self-affine sets, self-conformal sets) and many other sets including Mandelbrot percolation, Julia sets etc. Many of these sets may have Fourier dimension zero (arithmetic self-similar sets, self-affine sponges etc) but it turns out that even when the Fourier dimension is zero, the Fourier spectrum will be ‘as large as possible’ near $\theta = 0$.

LEMMA 2.5.5. *If $X \subset \mathbb{R}^d$ is a compact set with $\dim_{\mathbb{F}} X = 0$ which supports a (C, α) -uniformly affinely non-concentrated Borel measure, then*

$$\limsup_{\theta \rightarrow 0} \frac{\dim_{\mathbb{F}}^{\theta} X}{\theta} = d.$$

PROOF. Let μ be a (C, α) -uniformly affinely non-concentrated Borel measure on X . Then [67, Theorem 1.6] gives that the L^{∞} -dimension of the n -fold convolution μ^{*n} converges to d as $n \rightarrow \infty$. However, since $\dim_{\mathbb{F}} \mu \leq \dim_{\mathbb{F}} X = 0$, it follows from [37, Corollary 1.8] and the fact that the L^{∞} -dimension is a lower bound for the Sobolev dimension, that $\dim_{\mathbb{S}}(\mu^{*n}) \rightarrow d$ as $n \rightarrow \infty$ also. Moreover, by Lemma 1.5.1

$\dim_S(\mu^{*n}) = n \dim_F^{1/n} \mu$, so we get

$$\limsup_{\theta \rightarrow 0} \frac{\dim_F^\theta X}{\theta} \geq \limsup_{n \rightarrow \infty} \frac{\dim_F^{1/n} \mu}{1/n} = \limsup_{n \rightarrow \infty} \dim_S(\mu^{*n}) = d.$$

Recall that by Theorem 1.6.3, the reverse inequality also holds. $\square$

From Lemma 2.5.5 we get that for compact sets $X \subseteq \mathbb{R}^d$ with $0 < \dim_H X < d$ supporting a (C, α) -uniformly affinely non-concentrated Borel measure, $\dim_F^\theta X > \theta \max\{k, \dim_H X\}$ for all θ sufficiently small. Thus, there is a large family of sets that satisfy the hypothesis of Corollary 2.5.4.

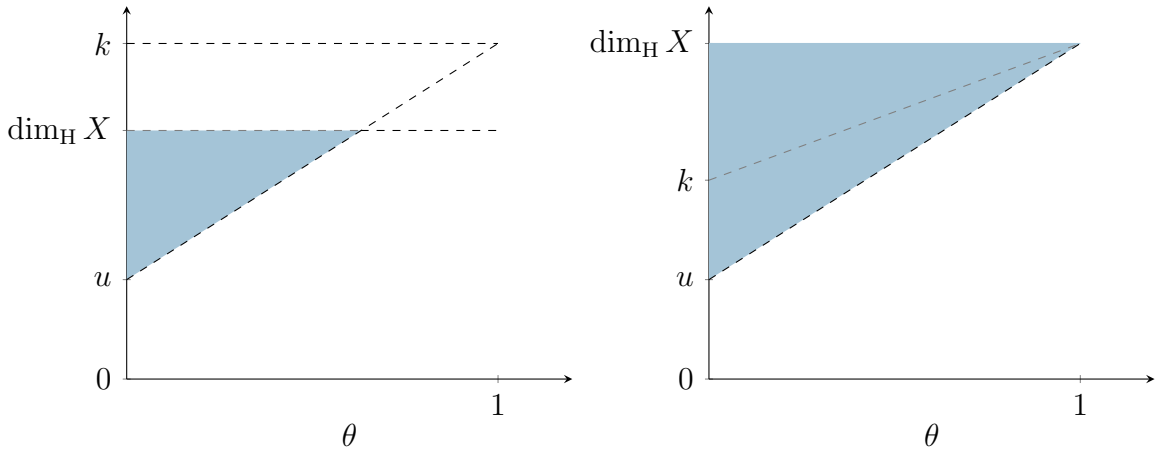


FIGURE 5. In order to improve the higher-dimensional exceptional set estimates for a Borel set $X \subseteq \mathbb{R}^d$ at u , we need the Fourier spectrum of X to intersect the shaded region i.e. for some $\theta \in [0, 1]$ the point $(\theta, \dim_F^\theta X)$ lies in the shaded region. Left: when $\dim_H X \leq k$. Right: when $\dim_H X > k$.

Finally, considering large θ , we obtain the following corollary taking the limit as $\theta \rightarrow 1$

COROLLARY 2.5.6. *Let $X \subseteq \mathbb{R}^d$ be a Borel set with $\dim_H X > k$ and*

$$D = \overline{\partial_-} \dim_F^\theta X \big|_{\theta=1} = \limsup_{\theta \rightarrow 1} \frac{\dim_H X - \dim_F^\theta X}{1 - \theta},$$

be the upper left semi-derivative of $\dim_F^\theta X$ at $\theta = 1$. If $D < \dim_H X - u$ for some $u \in [0, k]$ then there is necessarily improvement on Falconer's inequality (2.1.7).

Gan's bound (2.1.8) is always better than Mattila's (2.1.6) and only better than Falconer's (2.1.7) if $\dim_H X < \frac{dk}{d-k}$. Thus, it is only interesting to see when the bound obtained with the Fourier spectrum is better than Gan's bound for $\dim_H X \leq \frac{dk}{d-k}$. Setting $u = k \dim_H X / d$ in Corollary 2.5.1 we see that this is possible if for some

$\theta \in (0, 1]$,

$$(2.5.7) \quad \theta \min\{k, d - k\} + \frac{k \dim_{\mathbb{H}} X}{d} < \dim_{\mathbb{F}}^{\theta} X.$$

Thus, if $\dim_{\mathbb{H}} X$ is small and X supports a (C, α) -uniformly affinely non-concentrated measure, Lemma 2.5.5 shows that the spectrum will satisfy (2.5.7) for $\theta \in (0, 1]$ small enough depending on $\dim_{\mathbb{H}} X$.

2.5.2. Dimension bounds for the set of projections with empty interior.

Following [96, Corollary 6.2], we use the previous exceptional set estimate to obtain results regarding the interior of projections. Note first that if $\dim_{\mathbb{F}}^{\theta} X > 2k$ for some $\theta \in [0, 1]$, then $\dim_{\mathbb{H}} X > 2k$ and by [83, Corollary 5.12 (d)], $P_V(X)$ has non-empty interior for $\gamma_{d,k}$ almost all $V \in G(d, k)$.

COROLLARY 2.5.7. *Let $X \subseteq \mathbb{R}^d$ be a Borel set and $1 \leq k < d$ be an integer. If $\dim_{\mathbb{F}}^{\theta} X > 2k$ for some $\theta \in (0, 1]$, then*

$$\begin{aligned} \dim_{\mathbb{H}}\{V \in G(d, k) : P_V(X) \text{ has empty interior}\} &\leq k(d - k) + \inf_{\theta \in (0, 1]} \frac{2k - \dim_{\mathbb{F}}^{\theta} X}{\theta} \\ &< k(d - k). \end{aligned}$$

PROOF. Fix $\theta \in (0, 1]$, let $\varepsilon > 0$ and choose $\mu \in \mathcal{M}(X)$ such that $\dim_{\mathbb{F}}^{\theta} \mu \geq \dim_{\mathbb{F}}^{\theta} X - \varepsilon$. We will use the following inclusion of sets, which follows from the fact that if $\dim_{\mathbb{F}}^{\theta} \mu_V > 2k$, then μ_V is continuous (see [83, Theorem 5.4 (c)]) and $P_V(X)$ has non-empty interior. Therefore,

$$\begin{aligned} \{V \in G(d, k) : P_V(X) \text{ has empty interior}\} &\subseteq \{V \in G(d, k) : \mu_V \text{ not continuous}\} \\ &\subseteq \{V \in G(d, k) : \dim_{\mathbb{F}}^{\theta} \mu_V \leq 2k\} \\ &\subseteq \{V \in G(d, k) : \dim_{\mathbb{F}}^{\theta} \mu_V < 2k + \varepsilon\}. \end{aligned}$$

Theorem 2.4.1 gives an upper bound for the dimension of the final set in this chain of inclusions and therefore, since $\theta \in (0, 1]$ was arbitrary,

$$\begin{aligned} \dim_{\mathbb{H}}\{V \in G(d, k) : P_V(X) \text{ has empty interior}\} &\leq k(d - k) + \inf_{\theta \in (0, 1]} \frac{2k + \varepsilon - \dim_{\mathbb{F}}^{\theta} \mu}{\theta} \\ &\leq k(d - k) + \inf_{\theta \in (0, 1]} \frac{2k + 2\varepsilon - \dim_{\mathbb{F}}^{\theta} X}{\theta} \end{aligned}$$

and letting $\varepsilon \rightarrow 0$ proves the result. $\square$

In [96, Corollary 6.2] the authors proved that

$$(2.5.8) \quad \dim_{\mathbb{H}}\{V \in G(d, k) : P_V(X) \text{ has empty interior}\} \leq k(d - k) + 2k - \dim_{\mathbb{H}} X.$$

Thus, the Fourier spectrum improves on (2.5.8) if for some $\theta \in (0, 1]$,

$$\dim_{\mathbb{F}}^{\theta} X > 2k(1 - \theta) + \theta \dim_{\mathbb{H}} X,$$

which brings us back to the discussion at the end of Section 2.5.1.2 for $u = 2k$.

2.5.3. Continuity of the dimension of the exceptional set.

Proposition 2.2.2 showed us that for sets X , the dimension of the set of exceptional directions can be discontinuous at $u = \dim_{\mathbb{F}} X$. One of the questions which motivated our investigation in the first place was to determine conditions under which continuity of the dimension of the set of exceptional directions could be recovered at $u = \dim_{\mathbb{F}} X$. We show in the following proposition that such a condition can be given in terms of the Fourier spectrum. There is an analogous result for measures, which we leave to the reader to formulate.

PROPOSITION 2.5.8. *Let X be a Borel set in $\mathbb{R}^d$ and let*

$$D = \underline{\partial}_+ \dim_{\mathbb{F}}^{\theta} X|_{\theta=0} = \liminf_{\theta \rightarrow 0} \frac{\dim_{\mathbb{F}}^{\theta} X - \dim_{\mathbb{F}} X}{\theta}$$

be the lower right semi-derivative of $\dim_{\mathbb{F}}^{\theta} X$ at $\theta = 0$. If $D \geq k(d - k)$, then the function $u \mapsto \dim_{\mathbb{H}}\{V \in G(d, k) : \dim_{\mathbb{H}} P_V(X) < u\}$ is continuous at $u = \dim_{\mathbb{F}} X$.

PROOF. Let $\varepsilon \in (0, 1)$ and consider $u = \dim_{\mathbb{F}} X + \varepsilon^2$. Corollary 2.5.1 gives that

$$\begin{aligned} & \dim_{\mathbb{H}} \{V \in G(d, k) : \dim_{\mathbb{H}} P_V(X) < \dim_{\mathbb{F}} X + \varepsilon^2\} \\ & \leq \max \left\{ 0, k(d - k) + \inf_{\theta \in (0, 1]} \frac{\dim_{\mathbb{F}} X + \varepsilon^2 - \dim_{\mathbb{F}}^{\theta} X}{\theta} \right\} \\ & \leq \max \left\{ 0, k(d - k) + \varepsilon - \frac{\dim_{\mathbb{F}}^{\varepsilon} X - \dim_{\mathbb{F}} X}{\varepsilon} \right\} \\ & \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$ provided $D \geq k(d - k)$, which proves the desired continuity result. $\square$

We know by Theorem 1.6.3 that for any Borel set $X \subseteq \mathbb{R}^d$, $\underline{\partial}_+ \dim_{\mathbb{F}}^{\theta} X|_{\theta=0} \leq d$. Therefore, in order to establish continuity of the dimension of the exceptional set from Proposition 2.5.8, it is necessary to have $k(d - k) \leq d$. This is only true for the families $G(d, 1)$, $G(d, d - 1)$, and the special case $G(4, 2)$.

In Proposition 2.2.2 we built a non-Salem set $X \subseteq \mathbb{R}^2$ for which the dimension of the exceptional set was discontinuous at $\dim_{\mathbb{F}} X$. However, like Salem sets, the set X satisfies $\underline{\partial}_+ \dim_{\mathbb{F}}^{\theta} X|_{\theta=0} = 0$. To see this, recall that X was the union of a set A coming from Lemma 2.2.1 and a Salem set B . With a little more work, one can show that $\dim_{\mathbb{F}} A < \dim_{\mathbb{F}} B$. Thus, there must exist $\lambda \in (0, 1)$ such that $\dim_{\mathbb{F}}^{\lambda} A = \dim_{\mathbb{F}} B$ and

then for all $\theta \leq \lambda$, since $X = A \cup B$, $\dim_F^\theta X = \dim_F^\theta B$, which gives $\partial_+ \dim_F^\theta X|_{\theta=0} = 0$. This raises the following question.

QUESTION 2.5.9. *Is $\underline{\partial}_+ \dim_F^\theta X|_{\theta=0} > 0$ sufficient to guarantee continuity of the dimension of the exceptional set at $u = \dim_F X$? Or perhaps $\underline{\partial}_+ \dim_F^\theta X|_{\theta=0} \geq \rho$ for some $\rho > 0$? (We know from the above that $\rho = k(d - k)$ suffices.)*

2.6. An explicit example

Here we provide an explicit example where we exploit dynamical structure to get good information about the Fourier spectrum at $\theta = 1/2$. We feed this information into Corollary 2.5.1 and use this to showcase how our results compare with the best known general bounds in a representative situation. In general, computing the Fourier spectrum explicitly is difficult and so our example is by necessity quite simple. In particular, one may attempt to bound the dimension of the exceptional set directly using the specific structure at hand or by directly quoting the extensive literature on self-similar and self-affine sets—and probably obtain much stronger results—but we do not pursue this here. The example we consider is a self-affine set X in $\mathbb{R}^3$ built as the product of three self-similar sets in the line with different contraction ratios. We design the example such that the sumset $X + X$ is itself a self-affine set satisfying the open set condition. Then, for μ , a self-affine measure on X (of maximal Hausdorff dimension), we apply Lemma 1.5.1 to get

$$\dim_F^{1/2} \mu = \frac{\dim_S(\mu * \mu)}{2},$$

and we can compute the right-hand side explicitly since $\mu * \mu$ is again a self-affine measure and $\min\{\dim_S(\mu * \mu), 2\}$ coincides with the L^2 -dimension, for which there is an explicit formula (see Section 1.4). The Fourier spectrum then gives non-trivial information in the sense that it beats the trivial lower bound $\dim_F^\theta X \geq \theta \dim_H X$ provided $\dim_S(\mu * \mu) > \dim_H X$ (and this is indeed the case). This strategy can be employed in much greater generality than we give here, but we leave further examples to the reader; recall Corollary 2.5.2 for a general formulation. Better bounds would be obtained by considering smaller θ , but the Fourier spectrum becomes harder to estimate in this case.

Fix $\alpha, \beta, \gamma \in (0, 1/3]$ and let $E_\alpha, E_\beta, E_\gamma \subseteq [0, 1]$ denote the middle $(1 - 2\alpha)$, $(1 - 2\beta)$ and $(1 - 2\gamma)$ Cantor sets, respectively. Let $X = E_\alpha \times E_\beta \times E_\gamma$, which is a self-affine set of Hausdorff dimension

$$\dim_H X = \frac{\log 2}{-\log \alpha} + \frac{\log 2}{-\log \beta} + \frac{\log 2}{-\log \gamma}.$$

In many cases it is clear that $\dim_F X = 0$ (for example if at least one of α, β, γ is the reciprocal of a Pisot number, see [83, Theorem 8.3]) and we are unaware of any examples where it is known that $\dim_F X > 0$. Let μ be the uniform measure on X , that is, the self-affine measure defined with 8 equal weights. Then $\mu * \mu$ is a self-affine measure associated with a system of 27 maps (not with equal weights) and by standard results (or direct calculation)

$$\begin{aligned} \dim_S(\mu * \mu) &= \frac{\log((1/4)^2 + (1/2)^2 + (1/4)^2)}{\log \alpha} + \frac{\log((1/4)^2 + (1/2)^2 + (1/4)^2)}{\log \beta} \\ &\quad + \frac{\log((1/4)^2 + (1/2)^2 + (1/4)^2)}{\log \gamma} \\ &= \frac{\log(8/3)}{-\log \alpha} + \frac{\log(8/3)}{-\log \beta} + \frac{\log(8/3)}{-\log \gamma}. \end{aligned}$$

By Corollary 2.5.2 (setting $n = 2$) we get

$$\dim_H\{V \in G(3, k) : \dim_H P_V(X) < u\} \leq \max \left\{ 0, 2 + 2u - \frac{\log(8/3)}{-\log \alpha} - \frac{\log(8/3)}{-\log \beta} - \frac{\log(8/3)}{-\log \gamma} \right\}$$

for all $u \in [0, k]$ where $k \in \{1, 2\}$. See Figure 6 for a comparison of the above bound with Falconer's (2.1.7) and Mattila's (2.1.6) for the cases $k = 1$ and $k = 2$. Interesting to observe is the fact that in Figure 6 the bound given with the Fourier spectrum shows that dimension of the exceptional set has 0 Hausdorff dimension in a range of values of u for which the other two bounds give positive dimension.

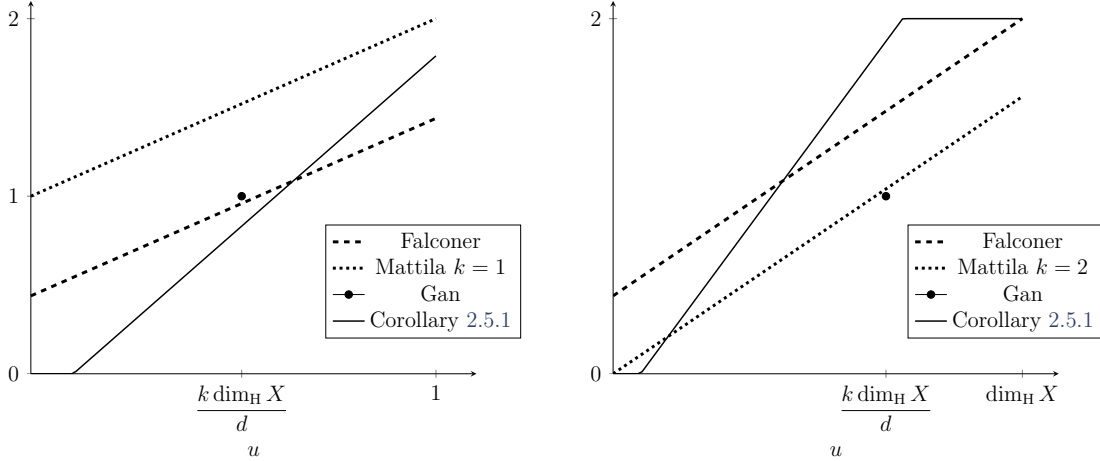


FIGURE 6. Exceptional set estimates given by the Fourier spectrum at $\theta = 1/2$, Gan's (2.1.8) at $u = k \dim_H X/d$, Falconer's (2.1.7) and Mattila's (2.1.6), for $\alpha = 1/3$, $\beta = 1/4$ and $\gamma = 1/5$, with $\dim_H X \approx 1.56$. Left: when $k = 1$. Right: when $k = 2$.

The key feature in the example above is the fact that $X + X$ is self-affine and satisfies the open set condition. Sharper bounds could be obtained considering n -fold

convolutions of sets for which nX also satisfies the open set condition, by Lemma 1.5.1 and using (1.4.1) to compute $\dim_S \mu^{*n}$. We note that this example is intentionally simple, and it is likely that for this specific family of sets, the dimension of the exceptional set could be computed more explicitly by exploiting the product structure directly or by appealing to the extensive literature on projections of self-affine sets. This would almost certainly yield sharper bounds than those obtained here with the Fourier spectrum. The purpose of this example is therefore not to obtain optimal bounds in this specific case, but rather to illustrate how the Fourier spectrum can be used to improve on general exceptional set estimates in a concrete and computable setting.

CHAPTER 3

The Fourier restriction problem

3.1. Introduction

The celebrated *Fourier restriction problem* asks when is it meaningful to restrict the Fourier transform of a function to a set of measure zero, such as a sphere, paraboloid, or cone. It is one of the central problems in harmonic analysis due to its connections to questions like the Bochner–Riesz conjecture, the Kakeya problem, decoupling, and the theory of dispersive PDEs.

More precisely, the Fourier restriction problem asks for which $p, q \in [1, \infty]$ does it hold that for a Borel measure μ ,

$$(3.1.1) \quad \|\widehat{f}\|_{L^{p'}(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

By duality of L^p spaces, (3.1.1) is equivalent to

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mu)}$$

with (3.1.1) referred to as an $L^{q'} \rightarrow L^{p'}$ restriction estimate and the dual form as an $L^p \rightarrow L^q$ extension estimate. What is more, if $p = 2$, both restriction and extension estimates are equivalent to

$$(3.1.2) \quad \|\widehat{\mu} * f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

We suppose from now on that the support of μ is a Lebesgue null set in $\mathbb{R}^d$ (and therefore we are genuinely attempting to restrict). If $q' = 1$ then f is integrable and $\widehat{f}$ is continuous and bounded, and so estimates of the form (3.1.1) will be possible for all p' . However, since the Fourier transform is an $L^2(\mathbb{R}^d)$ isometry, if $q' = 2$, $\widehat{f} \in L^2(\mathbb{R}^d)$ and is only defined Lebesgue almost everywhere and so (3.1.1) will not hold for any p' . Thus, the question is only interesting for the range $1 < q' < 2$.

Particular instances of this problem have become central questions in the field. For example, for the surface measure on both the sphere and paraboloid, the conjecture is that (3.1.1) holds if

$$(3.1.3) \quad \frac{d-1}{p'} \geq \frac{d+1}{q} \quad \text{and} \quad q > \frac{2d}{d-1},$$

(see [106, Chapter IX]) and is still open for $d \geq 3$. This conjecture has only been settled in the $d = 2$ case ([33, 123]), but significant progress has been made in higher dimensions over the last five decades (see e.g. [116] and the references therein). We refer the reader to [83], and [24] for a more thorough presentation of its history. One of the motivations behind the interest on restriction estimates for surfaces measures on manifolds is due to its applications to PDEs. For example, the extension operator on the paraboloid $\widehat{f\sigma_{\mathbb{P}}}$ is the solution to the free Schrödinger equation, which describes the evolution of the wave function of a free particle in quantum mechanics

$$(3.1.4) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) = 2\pi i \Delta u(x, t), & (x, t) \in \mathbb{R}^{d-1} \times \mathbb{R}; \\ u(x, 0) = g(x), & x \in \mathbb{R}^{d-1}. \end{cases}$$

Here $\sigma_{\mathbb{P}}$ is the surface measure on the paraboloid and $f = \check{g}$. If the Fourier extension estimate (3.1.2) holds for the paraboloid and $\check{g} \in L^p(\sigma_{\mathbb{P}})$ for some $p \in [1, \infty]$, then $u \in L^q(\mathbb{R}^d)$ where $p, q \in [1, \infty]$ satisfy (3.1.3).

3.1.1. Stein–Tomas restriction.

Stein observed that non-trivial estimates are often possible when μ is the surface measure on a smooth curved manifold, but that nothing can be said when the surface has flat pieces. In general, such curvature features can be captured by Fourier decay and this led to, over many years, the Stein–Tomas restriction theorem. The most general version of this is due to Bak–Seeger [1], but builds on work of Stein (see [106]), Tomas [113], Mockenhaupt [85], Mitsis [84], and others (see also [33, 13]). This theorem considers the case $p = 2$ and gives a non-trivial range of q for which (3.1.1) holds in terms of the Frostman exponent and Fourier decay of μ .

THEOREM 3.1.1 (Stein–Tomas). *Let μ be a finite, compactly supported, Borel measure on $\mathbb{R}^d$ such that for some $\alpha, \beta > 0$,*

$$(3.1.5) \quad \mu(B(x, r)) \lesssim r^\alpha$$

for all $x \in \mathbb{R}^d$ and $r > 0$, and

$$(3.1.6) \quad |\widehat{\mu}(\xi)|^2 \lesssim |\xi|^{-\beta}$$

for all $\xi \in \mathbb{R}^d$. Then for all $f \in L^2(\mu)$, the estimate

$$(3.1.7) \quad \|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$$

holds for all

$$(3.1.8) \quad q \geq 2 + 4 \frac{d - \alpha}{\beta}.$$

By taking appropriate limits, Theorem 3.1.1 implies that (3.1.7) holds for all

$$q > 2 + 4 \frac{d - D_\infty(\mu)}{\dim_F \mu}.$$

This version—stated in terms of dimensions—is useful to keep in mind when comparing with our main theorem. Throughout this Thesis, we shall refer to (3.1.7) with (3.1.8) as the Stein–Tomas estimate.

This distinction between Fourier decay (as with curved manifolds) and no Fourier decay (as with flat manifolds) is also fundamental in the fractal geometry literature. Random sets often exhibit Fourier decay (see e.g. [101]), whereas sets with arithmetic structure often do not (see e.g. [78]). Thus, the question asked by Stein naturally extends to fractals and suggests a large programme of research investigating restriction problems for fractal measures in various contexts.

In this Chapter we obtain a new range of q for which (3.1.7) holds in terms of the Fourier spectrum (see Section 1.5.2). In Section 3.2 we state and prove the main theorem of this Chapter (Theorem 3.2.1), which follows by an interpolation argument between the $L^2 \rightarrow L^2$ estimate given in [85] using the L^∞ -dimension and a new $L^{q'} \rightarrow L^q$ estimate obtained directly from the Fourier spectrum, which we state precisely in Corollary 3.2.4.

Theorem 3.2.1 provides a continuum of Stein–Tomas type estimates, which are optimised to provide our main result. However, it is worth noting that at one end of this continuum we recover the usual Stein–Tomas theorem and at the other end we obtain a new estimate which can be stated in terms of the Fourier and *Sobolev* dimensions which typically outperforms Stein–Tomas for multifractal measures, see Corollary 3.5.1. We also provide results in the other direction by giving a range of q in terms of the Fourier spectrum for which the $L^2 \rightarrow L^q$ extension estimate (3.1.7) fails, generalising an observation of Hambrook and Łaba (see Theorem 3.2.6). In Section 3.3 we give a restriction estimate for Lorentz spaces using the Fourier spectrum, which allows us to prove the endpoint estimate Theorem 3.2.2, of the main theorem (Theorem 3.2.1). We illustrate our results with several examples, including the surface measure on the cone and the moment curve and several fractal measures (see Sections 3.6–3.8).

In order to apply our results to the surface measure on the cone, we first explicitly derive the Fourier spectrum for this measure (see Proposition 3.6.2). We also give the Fourier spectrum of arclength measure on the moment curve (see Proposition 3.7.1), and this example exhibits multiple phase transitions which has not been previously observed in ‘natural examples’.

3.2. Main results

3.2.1. L^2 restriction estimates from the Fourier spectrum.

In the following theorem we give a new range of q for the $L^{q'} \rightarrow L^2$ Fourier restriction estimate to hold. The proof follows that of [85] where instead of using the Fourier dimension of μ , we use its Fourier spectrum, which allows us to interpolate between the known $L^2 \rightarrow L^2$ estimate given by the Frostman condition with a new $L^{q'} \rightarrow L^q$ estimate given by the Fourier spectrum, where we use the information given by the (s, θ) -energies of a discretisation of μ at different scales.

THEOREM 3.2.1. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If*

$$q > 2 + 2 \inf_{\substack{\theta \in [0,1] \\ \dim_{\mathbb{F}}^{\theta} \mu > d\theta}} \frac{(d - D_{\infty}(\mu))(2 - \theta)}{\dim_{\mathbb{F}}^{\theta} \mu - D_{\infty}(\mu)\theta},$$

then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, if

$$1 \leq q' < 1 + \sup_{\substack{\theta \in [0,1] \\ \dim_{\mathbb{F}}^{\theta} \mu > d\theta}} \frac{\dim_{\mathbb{F}}^{\theta} \mu - D_{\infty}(\mu)\theta}{\dim_{\mathbb{F}}^{\theta} \mu + 4(d - D_{\infty}(\mu)) - \theta(2d - D_{\infty}(\mu))},$$

then for all $f \in L^{q'}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

In fact, we can obtain the appropriate endpoint estimate. Although the following theorem implies Theorem 3.2.1, its proof is much more involved, since it depends on restriction estimates for Lorentz spaces. For this reason we postpone its proof to Section 3.3, and give a direct proof of Theorem 3.2.1 at the end of this section.

THEOREM 3.2.2. *Let $0 < \alpha < d$, $\theta \in [0, 1]$, $d\theta < s$, and let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$, such that*

$$\mu(B(x, r)) \lesssim r^{\alpha}, \quad \forall x \in \mathbb{R}^d, r > 0;$$

and

$$\mathcal{J}_{s,\theta}(\mu) < \infty.$$

Let

$$q_0 = 2 + 2 \frac{(d - \alpha)(2 - \theta)}{s - \alpha\theta}.$$

Then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^{q_0}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, let

$$q'_0 = 1 + \frac{s - \alpha\theta}{s + 4(d - \alpha) - \theta(2d - \alpha)}.$$

Then for all $f \in L^{q'_0}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{q'_0}(\mathbb{R}^d)}.$$

The conditions $\dim_{\mathbb{F}}^{\theta} \mu > d\theta$ and $s > d\theta$ are necessary when taking the infimum in Theorem 3.2.1 and in Theorem 3.2.2. Indeed, the middle third Cantor measure has $\dim_{\mathbb{F}}^{\theta} \mu \leq \theta$ for all θ (see [37, Corollary 4.4]), but if we attempt to apply Theorem 3.2.1 or Theorem 3.2.2 without the condition we would get a non-trivial restriction range because $\dim_{\mathbb{F}}^{1/2} \mu > \alpha/2$ (see [37, Corollary 4.4]). However, this is not possible by the following lemma, described to us by Chun-Kit Lai. In particular, the Fourier transform of the middle third Cantor measure does not decay to 0 at ∞ .

LEMMA 3.2.3. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If for some $q > 1$, $\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \leq \|f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$, then $|\widehat{\mu}(\xi)| \rightarrow 0$ as $|\xi| \rightarrow \infty$.*

PROOF. Taking $f = 1$, and using that μ has compact support, $f \in L^2(\mu)$ and so $\widehat{\mu} \in L^q(\mathbb{R}^d)$. However, since $\widehat{\mu}$ is uniformly continuous (even uniformly Lipschitz, see [83, (3.19)]) this implies that $\widehat{\mu}$ must decay at infinity. $\square$

In the proof of Theorem 3.2.1 (see Section 3.2.1.1) we obtained the following restriction estimate directly from the Fourier spectrum, without appealing to the $L^2 \rightarrow L^2$ estimate coming from the Frostman exponent. We record it here for interest.

COROLLARY 3.2.4. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If*

$$q > \inf \left\{ \frac{4}{\theta} : \theta \in [0, 1], \dim_{\mathbb{F}}^{\theta} \mu > d\theta \right\},$$

then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, if

$$1 \leq q' < \sup \left\{ \frac{4}{4 - \theta} : \theta \in [0, 1], \dim_{\mathbb{F}}^{\theta} \mu > d\theta \right\},$$

then for all $f \in L^{q'}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

As a simple consequence of Theorem 3.2.2 and real interpolation with the trivial $L^1 \rightarrow L^\infty$ estimates

$$\|\widehat{f\mu}\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mu)} \quad \text{and} \quad \|\widehat{f}\|_{L^\infty(\mu)} \lesssim \|f\|_{L^1(\mathbb{R}^d)},$$

we obtain the following corollary.

COROLLARY 3.2.5. *Let $0 < \alpha < d$, $\theta \in [0, 1]$, $d\theta < s$, and let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$, such that*

$$\mu(B(x, r)) \lesssim r^\alpha, \quad \forall x \in \mathbb{R}^d, r > 0;$$

and

$$\mathcal{J}_{s, \theta}(\mu) < \infty.$$

For $q \geq q_0$, where q_0 is as defined in Theorem 3.2.2, the estimate

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{2q/(2q-q_0)}(\mu)}$$

holds for all $f \in L^{\frac{2q}{2q-q_0}}(\mu)$.

Equivalently, for $1 \leq q' \leq q'_0$,

$$\|\widehat{f}\|_{L^{2q/q_0}(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}$$

holds for all $f \in L^{q'}(\mathbb{R}^d)$.

L^2 restriction theorems of the type proved in this section have numerous applications. For example, in [85, Theorem 5.1] Mockenhaupt uses the Stein–Tomas theorem to prove that certain convolution operators are L^p -multipliers, with the range of p depending on the established range for L^2 restriction. One can do the same using our Theorem 3.2.1 and obtain improved ranges, but we leave the details to the reader.

3.2.1.1. Proof of Theorem 3.2.1.

We will show that $\|\widehat{\mu} * f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}$ for every $f \in L^{q'}(\mathbb{R}^d)$ for all q sufficiently close to the desired threshold. This is equivalent to both statements in the theorem (see Section 3.1). In particular, establishing (3.1.7) for some q establishes it for all larger q since μ is compactly supported.

To begin with, let $q \geq 1$ be arbitrary. We start by breaking up $\widehat{\mu}$ dyadically in the standard way. Let $\rho \in \mathcal{C}^\infty(\mathbb{R}^d)$ be such that $\rho(\xi) = 0$ for $|\xi| \leq 1/2$ and $\rho(\xi) = 1$ for $|\xi| \geq 1$. Define the function $\varphi(\xi) = \rho(2\xi) - \rho(\xi)$, noting that $\text{spt}\varphi \subset \{\xi \in \mathbb{R}^d : \frac{1}{4} \leq |\xi| \leq 1\}$ and

$$\sum_{j=0}^{\infty} \varphi(2^{-j}\xi) = 1 \quad (\text{for } |\xi| \geq 1).$$

Clearly $\widehat{\mu} = \Phi + \sum_{j=0}^{\infty} \widehat{\mu}_j$ where

$$\Phi(\xi) = \left(1 - \sum_{j=0}^{\infty} \varphi(2^{-j}\xi)\right) \widehat{\mu}(\xi)$$

and $\widehat{\mu}_j(\xi) = \varphi(2^{-j}\xi)\widehat{\mu}(\xi)$. Therefore,

$$(3.2.1) \quad \|\widehat{\mu} * f\|_{L^q(\mathbb{R}^d)} \leq \|\Phi * f\|_{L^q(\mathbb{R}^d)} + \sum_{j=0}^{\infty} \|\widehat{\mu}_j * f\|_{L^q(\mathbb{R}^d)}.$$

For the first term, since Φ is bounded and has compact support,

$$\|\Phi\|_{L^{q/2}(\mathbb{R}^d)} \lesssim \|\Phi\|_{L^\infty(\mathbb{R}^d)} \lesssim 1.$$

Using this and Young's inequality for convolutions gives,

$$\|\Phi * f\|_{L^q(\mathbb{R}^d)} \leq \|\Phi\|_{L^{q/2}(\mathbb{R}^d)} \|f\|_{L^{q'}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

For the second term we will obtain the $L^{q'} \rightarrow L^q$ bound by interpolation. Let $\theta \in (0, 1]$ and $s > 0$ be such that $\dim_{\mathbb{F}}^\theta \mu > s > d\theta$. Since $\mathcal{J}_{s,\theta}(\mu) \lesssim 1$,

$$\|\widehat{\mu}_j\|_{L^{2/\theta}(\mathbb{R}^d)} \approx \left(\int_{|\xi| \approx 2^j} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-d} 2^{-j(\frac{s}{\theta}-d)} d\xi \right)^{\frac{\theta}{2}} \lesssim 2^{-j\frac{s-d\theta}{2}} \mathcal{J}_{s,\theta}(\mu)^{1/2} \lesssim 2^{-j\frac{s-d\theta}{2}},$$

and so by Young's inequality for convolutions,

$$(3.2.2) \quad \|\widehat{\mu}_j * f\|_{L^{4/\theta}(\mathbb{R}^d)} \leq \|\widehat{\mu}_j\|_{L^{2/\theta}(\mathbb{R}^d)} \|f\|_{L^{4/(4-\theta)}(\mathbb{R}^d)} \lesssim 2^{-j\frac{s-d\theta}{2}} \|f\|_{L^{4/(4-\theta)}(\mathbb{R}^d)}.$$

Following [85, Theorem 4.1] we can use the fact that μ is α_0 -Frostman for $\alpha_0 < \alpha$ to prove that

$$(3.2.3) \quad \|\widehat{\mu}_j * f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-j(\alpha_0-d)} \|f\|_{L^2(\mathbb{R}^d)}.$$

Let $\lambda \in [0, 1]$ and apply the Riesz–Thorin interpolation theorem with parameter λ between (3.2.2) and (3.2.3). This gives a $L^{q'} \rightarrow L^q$ estimate, with

$$\frac{1}{q} = \frac{(1-\lambda)\theta}{4} + \frac{\lambda}{2},$$

noting that this forces $2 \leq q \leq 4/\theta$. More precisely, we obtain

$$\|\widehat{\mu}_j * f\|_{L^q(\mathbb{R}^d)} \leq 2^{-j\left((1-\lambda)\frac{s-d\theta}{2} + \lambda(\alpha_0-d)\right)} \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

Therefore, provided

$$(3.2.4) \quad (1-\lambda)\frac{s-d\theta}{2} + \lambda(\alpha_0-d) > 0,$$

we get

$$\sum_{j=0}^{\infty} \|\widehat{\mu}_j * f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

Using this in (3.2.1) establishes the desired $L^{q'} \rightarrow L^q$ bound whenever $q \in [2, 4/\theta]$ satisfies (3.2.4), that is, whenever

$$\frac{4}{\theta} \geq q > 2 + 2\frac{(d-\alpha_0)(2-\theta)}{s-\alpha_0\theta}.$$

Crucially, using that $s > d\theta$ and $\alpha_0 < d$,

$$\frac{4}{\theta} > 2 + 2 \frac{(d - \alpha_0)(2 - \theta)}{s - \alpha_0\theta}$$

and so there is a non-empty range of possible q . This proves the desired result upon letting $\alpha_0 \rightarrow D_\infty(\mu)$ and $s \rightarrow \dim_F^\theta \mu$.

3.2.2. A partial converse from the Fourier spectrum.

Another interesting question is to determine when the restriction estimate (3.1.1) does not hold. In [50] the authors noted that (3.1.1) will not hold for any $2 \leq q < \frac{2d}{\dim_S \mu}$, which is a non-trivial statement when $\dim_S \mu < d$. We repeat their simple argument here. Let $d > s > \dim_S \mu$, in which case $I_s(\mu) = \infty$. Then, by Hölder's inequality with conjugate exponents $\frac{q}{2}$ and $\frac{q}{q-2}$,

$$(3.2.5) \quad \infty = \int_{|\xi| \geq 1} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi \leq \|\widehat{\mu}\|_q^2 \left(\int_{|\xi| \geq 1} |\xi|^{\frac{q(s-d)}{q-2}} d\xi \right)^{\frac{q-2}{q}},$$

and the last integral is finite if $q < \frac{2d}{s}$. Therefore, for $q < \frac{2d}{\dim_S \mu}$, $\widehat{\mu}$ cannot be in L^q and so (3.1.1) does not hold for any p . We can generalise this result (which we refer to later as the Hambrook–Łaba lower bound) using the Fourier spectrum.

THEOREM 3.2.6. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If $q = \frac{2}{\theta}$ for some $\theta \in (0, 1]$ such that $\mathcal{J}_{d\theta, \theta}(\mu) = \infty$ and $p \in [1, \infty]$, then for $f = 1$,*

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} = \infty > \|f\|_{L^p(\mu)} = \mu(\mathbb{R}^d)^{1/p}.$$

Hence, (3.1.1) does not hold for any $p \in [1, \infty]$.

In particular, (3.1.1) does not hold for any $p \in [1, \infty]$ if

$$q < \sup \left\{ \frac{2}{\theta} : \dim_F^\theta \mu < d\theta \right\}.$$

PROOF. Let $f = 1$ and $\theta \in (0, 1]$ be such that $\mathcal{J}_{d\theta, \theta}(\mu) = \infty$. Then

$$\infty = \mathcal{J}_{d\theta, \theta}(\mu) = \left(\int |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} d\xi \right)^\theta = \|\widehat{f\mu}\|_{L^{2/\theta}(\mathbb{R}^d)}^2,$$

as required. □

Note that for all $\theta \in (0, 1]$ such that $\dim_F^\theta \mu < d\theta$, $\mathcal{J}_{d\theta, \theta}(\mu) = \infty$. If $\dim_S \mu = \dim_F^1 \mu < d$, then we recover the fact that (3.1.1) fails for $q < \frac{2d}{\dim_S \mu}$. However, we will often get a stronger result (see Figure 9). Let us consider the smallest possible range provided by the previous theorem. This will happen when the Fourier spectrum stays above $d\theta$ for as large θ as possible. Since the Fourier spectrum is increasing in θ , this will happen when $\dim_F^\theta \mu = \dim_S \mu$ for $\theta \geq \dim_S \mu / d$. For non-Salem measures μ this

is quite unlikely to happen, and so we will obtain a strictly better (negative) range. On the other hand, the best information we can get from the above result is when the Fourier spectrum is as small as possible. Since the Fourier spectrum is concave this will happen when $\dim_F^\theta \mu$ is affine and given by

$$\dim_F^\theta \mu = \dim_F \mu + \theta(\dim_S \mu - \dim_F \mu).$$

In this case we get that (3.1.1) fails for

$$q < 2 + \frac{2(d - \dim_S \mu)}{\dim_F \mu}.$$

Theorem 3.2.6 seems to provide useful new information concerning the general restriction problem. For example, for the sphere, it gives that (3.1.1) fails for

$$q < \frac{2d}{d-1}$$

and this is one of the sharp thresholds from the restriction conjecture. This threshold is also obtained by the Hambrook–Łaba bound. However, we will see this behaviour again later when we consider the cone and the moment curve and, in fact, Theorem 3.2.6 gives a sharp threshold for the general restriction problem for these examples whereas the Hambrook–Łaba bound does not (see Sections 3.6 and 3.7).

Another simple consequence of the above is the following mild strengthening of Lemma 3.2.3.

COROLLARY 3.2.7. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If $\mathcal{J}_{d\theta, \theta}(\mu) = \infty$ for all $\theta \in (0, 1]$, then (3.1.1) does not hold for any pairs $p, q \in [1, \infty]$ with $q < \infty$. In particular, this is true if $\dim_F^\theta \mu < d\theta$ for all $\theta \in (0, 1]$ or if $\hat{\mu}$ does not decay at infinity.*

3.3. Restriction estimates for Lorentz spaces and the endpoint

3.3.1. Lorentz spaces and interpolation.

Lorentz spaces arise in the interpolation between L^1 and L^∞ , and provide a more nuanced description of the behaviour of functions and their distributions. For this reason, they are useful to obtain sharper inequalities involving L^p norms, leading to improvements on classical results such as the Hausdorff–Young inequality, and the Sobolev embedding theorem, among others (see [106, Chapter VI S7.10]). In particular, they have been useful to obtain improved $L^{q'} \rightarrow L^{p'}$ restriction estimates (see e.g. [58, 1, 122, 66]).

In this section we introduce restriction estimates between Lorentz spaces as a means to prove Theorem 3.2.2. For more details on Lorentz spaces we refer the reader to [48, 107].

Recall that for operators T acting on Lorentz spaces, by density of simple functions and the fact that $L^{q,\infty}$ is a normed space, being of restricted weak type (p, q) is equivalent to the fact that T maps $L^{p,1}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$ boundedly.

Given two quasinormed spaces compatible in the sense of interpolation theory (see [107, Chapter V]), and $\lambda \in [0, 1]$, we write $B_{\lambda,\infty}(B_0, B_1)$ to denote the real interpolation space between B_0 and B_1 . We shall only make use of this when $B_0 = L^p(\mathbb{R}^d)$ and $B_1 = L^q(\mathbb{R}^d)$, for some $1 \leq p, q \leq \infty$, in which case $B_{\lambda,\infty}(B_0, B_1) = L^{r,\infty}(\mathbb{R}^d)$ for $\frac{1}{r} = \frac{1-\lambda}{p} + \frac{\lambda}{q}$. The following lemma is an interpolation theorem stated implicitly by Bourgain in [9], which can also be found in [12, Section 6.2] in a more general setting. We state it here for convenience, in the form that we will use.

LEMMA 3.3.1 (Bourgain's interpolation trick). *Let A_0, A_1 , and B_0, B_1 be two pairs of quasinormed spaces compatible in the sense of interpolation theory. If T_j is a sequence of operators that satisfy that there exist constants $\alpha_0 < 0 < \alpha_1$, and $M_0, M_1 > 0$ such that for all $j \in \mathbb{N}$, $i = 0, 1$, and $f \in A_0 \cap A_1$,*

$$\|T_j f\|_{B_i} \leq M_i 2^{j\alpha_i} \|f\|_{A_i}.$$

Then for $\lambda = \frac{\alpha_0}{\alpha_0 - \alpha_1} \in (0, 1)$, and all $f \in A_0 \cap A_1$, the operator $T = \sum_{j=1}^{\infty} T_j$ satisfies

$$\|Tf\|_{B_{\lambda,\infty}(B_0, B_1)} \lesssim_{\alpha_0, \alpha_1} M_0^{1-\lambda} M_1^{\lambda} \|f\|_{A_0}^{1-\lambda} \|f\|_{A_1}^{\lambda}.$$

3.3.2. Restriction estimates for Lorentz spaces.

When considering restriction estimates in Lorentz spaces, the question becomes for which $p, q, r, s \in [1, \infty]$ does it hold that

$$(3.3.1) \quad \|\widehat{f}\|_{L^{r',s'}(\mu)} \lesssim \|f\|_{L^{p',q'}(\mathbb{R}^d)},$$

for all $f \in L^{p',q'}(\mathbb{R}^d)$. Similarly to the L^p case (see Section 3.1), by the dual characterisation of Lorentz quasinorms, the $L^{p',q'} \rightarrow L^{r',s'}$ restriction estimate (3.3.1) is equivalent to the $L^{r,s} \rightarrow L^{p,q}$ extension estimate

$$(3.3.2) \quad \|\widehat{f\mu}\|_{L^{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{L^{r,s}(\mu)}.$$

What is more, in the case $r = s = 2$, both (3.3.1) and (3.3.2) are equivalent to

$$\|\widehat{\mu} * f\|_{L^{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p',q'}(\mathbb{R}^d)}.$$

Theorem 3.2.2 is a consequence of the following restriction estimate for Lorentz spaces, which is also of independent interest. Note that when setting $\theta = 0$ in the following proposition, the constants ρ and σ are equal to those given in [1, Proposition 2.1].

PROPOSITION 3.3.2. *Let $0 < \alpha < d$, $\theta \in [0, 1]$, $d\theta < s$, and let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$, such that*

$$\mu(B(x, r)) \lesssim r^\alpha, \quad \forall x \in \mathbb{R}^d, r > 0;$$

and

$$\mathcal{J}_{s,\theta}(\mu) < \infty.$$

Let

$$\rho = \frac{4(2\alpha - s + d(\theta - 2))(\alpha - s + d(\theta - 1))}{2s^2 - 2\alpha^2(\theta - 4) - ds(7\theta - 12) - d^2(14\theta - 5\theta^2 - 8) - \alpha(\theta - 4)(-3s + d(3\theta - 4))};$$

$$\sigma = \frac{4(\alpha - s + d(\theta - 1))}{\theta(d + \alpha) - 2s}.$$

Then for any $\rho < p < \sigma'$, and $r > 0$ with $\frac{1}{p} - \frac{1}{r} = \frac{(d-\alpha)(2-\theta)}{2(d-\alpha)+s-d\theta}$, and $q \in [1, \infty]$,

$$(3.3.3) \quad \|\widehat{\mu} * f\|_{L^{r,q}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p,q}(\mathbb{R}^d)}.$$

What is more, (3.3.3) holds for $r = q_0$ and $p = q'_0$, where q_0 is as in Theorem 3.2.2.

3.3.3. Proof of Theorem 3.2.2.

First, by Parseval's formula,

$$(3.3.4) \quad \|\widehat{f}\|_{L^2(\mu)}^2 = \|\widehat{\bar{f}}\|_{L^2(\mu)}^2 = \int_{\mathbb{R}^d} \bar{f}(\xi)(\widehat{\mu} * f)(\xi) d\xi.$$

Hölder's inequality for Lorentz spaces with $1 = \frac{1}{q'_0} + \frac{1}{q_0}$ and $1 = \frac{1}{2} + \frac{1}{2}$, and the final part of Proposition 3.3.2 with $q = 2$, $r = q_0$, and $p = q'_0$ gives

$$\|\widehat{f}\|_{L^2(\mu)}^2 = \|\widehat{\bar{f}}\|_{L^{2,2}(\mu)}^2 \leq \|f\|_{L^{q'_0,2}(\mathbb{R}^d)} \|f * \widehat{\mu}\|_{L^{q_0,2}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'_0,2}(\mathbb{R}^d)}^2.$$

Since $q'_0 < 2$, $\|f\|_{L^{q'_0,2}(\mathbb{R}^d)} \leq \|f\|_{L^{q'_0}(\mathbb{R}^d)}$, and the result follows.

3.3.4. Proof of Proposition 3.3.2.

Let $\theta \in (0, 1]$ and $s > d\theta$ be such that $\mathcal{J}_{s,\theta}(\mu) < \infty$. By following the proof of Theorem 3.2.1 we get the following estimates for the dyadic decomposition $(\mu_j)_j$ of μ (see (3.2.2) and (3.2.3)):

$$(3.3.5) \quad \|\widehat{\mu}_j * f\|_{L^{4/\theta}(\mathbb{R}^d)} \lesssim 2^{-j\frac{s-d\theta}{2}} \|f\|_{L^{4/(4-\theta)}(\mathbb{R}^d)};$$

and

$$(3.3.6) \quad \|\widehat{\mu}_j * f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{j(d-\alpha)} \|f\|_{L^2(\mathbb{R}^d)}.$$

Since $s > d\theta$, we can use Bourgain's interpolation trick between the estimates (3.3.5) and (3.3.6). For this define

$$\lambda = \frac{s - d\theta}{2(d - \alpha) + s - d\theta}.$$

Let

$$q_0 = 2 + 2 \frac{(d - \alpha)(2 - \theta)}{s - \alpha\theta}$$

noting that $\frac{(1-\lambda)\theta}{4} + \frac{\lambda}{2} = \frac{1}{q_0}$, implies that $B_{\lambda,\infty}(L^{4/\theta}(\mathbb{R}^d), L^2(\mathbb{R}^d)) = L^{q_0,\infty}(\mathbb{R}^d)$. Hence, applying Lemma 3.3.1 with $f = 1_E$ for an arbitrary measurable set E ,

$$\|f\|_{L^{4/(4-\theta)}(\mathbb{R}^d)}^{1-\lambda} \|f\|_{L^2(\mathbb{R}^d)}^\lambda = |E|^{\frac{1}{q_0}} = \|f\|_{L^{q_0,1}(\mathbb{R}^d)},$$

thus, the convolution operator with $\widehat{\mu}$ is of restricted weak type (q'_0, q_0) . Thus, for all $f \in L^{q'_0,1}(\mathbb{R}^d)$,

$$(3.3.7) \quad \|\widehat{\mu} * f\|_{L^{q_0,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'_0,1}(\mathbb{R}^d)}.$$

The second restricted weak type estimate comes from [1, Proof of Proposition 2.1], where the authors use a special case of (3.3.7) to prove that

$$(3.3.8) \quad \|\widehat{\mu}_j * f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{j\frac{d-\alpha}{2}} \|f\|_{L^{q'_0,1}(\mathbb{R}^d)}.$$

We now use Bourgain's interpolation trick between (3.3.5) and (3.3.8). Let

$$\gamma = \frac{s - d\theta}{d - \alpha + s - d\theta}.$$

Note that for ρ and σ as defined in the proposition, $\frac{1}{\rho} = \frac{(1-\gamma)(4-\theta)}{4} + \frac{\gamma}{q'_0}$, and $\frac{1}{\sigma} = \frac{(1-\gamma)\theta}{4} + \frac{\gamma}{2}$. Thus, $B_{\gamma,\infty}(L^{4/\theta}(\mathbb{R}^d), L^2(\mathbb{R}^d)) = L^{\sigma,\infty}(\mathbb{R}^d)$, and letting $f = 1_E$ for an arbitrary measurable set E ,

$$\|f\|_{L^{4/(4-\theta)}(\mathbb{R}^d)}^{1-\gamma} \|f\|_{L^{q'_0,1}(\mathbb{R}^d)}^\gamma = |E|^{\frac{1}{\rho}} = \|f\|_{L^{\rho,1}(\mathbb{R}^d)},$$

which proves that the convolution operator with $\widehat{\mu}$ is of restricted weak type (ρ, σ) . That is, for all $f \in L^{\rho,1}(\mathbb{R}^d)$,

$$(3.3.9) \quad \|\widehat{\mu} * f\|_{L^{\sigma,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\rho,1}(\mathbb{R}^d)}.$$

Finally, using [48, Theorem 1.4.19] to interpolate between (3.3.9) and its dual

$$\|\widehat{\mu} * f\|_{L^{\rho',\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\sigma',1}(\mathbb{R}^d)},$$

we get that for any $\rho < p < \sigma'$, $\frac{1}{p} - \frac{1}{r} = \frac{(d-\alpha)(2-\theta)}{2(d-\alpha)+s-d\theta}$, and $q \in [1, \infty]$,

$$\|\widehat{\mu} * f\|_{L^{r,q}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p,q}(\mathbb{R}^d)},$$

as we wanted to prove. The final part of the result follows from this last interpolation by taking the interpolation parameter in [48, Theorem 1.4.19] to be $\frac{1}{2}$, since $\frac{1}{q_0}$ is the midpoint between $\frac{1}{\sigma}$ and $\frac{1}{\rho'}$, and $\frac{1}{q_0}$ the midpoint between $\frac{1}{\rho}$ and $\frac{1}{\sigma'}$.

3.4. Improving the Stein–Tomas range

In [50, 51, 19] the authors proved that there is a large family of measures satisfying (3.1.5) and (3.1.6), for which the range given by Stein–Tomas (3.1.8) is optimal. However, from the constructions of Chen [18], Chen–Seeger [20], Shmerkin–Suomala [103], and Łaba–Wang [72], we know that there are other measures (some also satisfying (3.1.5) and (3.1.6)) for which the range of q such that (3.1.7) holds can be improved. We now give a condition on the Fourier spectrum of measures for which the range given by Stein–Tomas is not optimal; see Figure 7.

The range of q obtained in Theorem 3.2.1 will improve on the Stein–Tomas range (3.1.8) if for some $\theta \in [0, 1]$

$$2 + \frac{2(d - D_\infty(\mu))(2 - \theta)}{\dim_F^\theta \mu - D_\infty(\mu)\theta} < 2 + \frac{4(d - D_\infty(\mu))}{\dim_F \mu}$$

and $\dim_F^\theta \mu > d\theta$. Therefore, we need

$$(3.4.1) \quad \dim_F^\theta \mu > \max \left\{ \theta \left(D_\infty(\mu) - \frac{\dim_F \mu}{2} \right) + \dim_F \mu, d\theta \right\}.$$

We note that by concavity $\dim_F^\theta \mu \geq (\dim_S \mu - \dim_F \mu)\theta + \dim_F \mu$ always holds.

3.5. A Stein–Tomas restriction theorem for the Sobolev dimension

If $\dim_F \mu > 0$, then there is an interval of θ for which $\dim_F^\theta \mu > d\theta$ (within which we can apply our estimate from Theorems 3.2.1 and 3.2.2). If we fix $\theta = 0$, then we (unsurprisingly) obtain the Stein–Tomas range. However, it is interesting to examine what happens at the other end, that is, Corollary 3.2.4. If we apply Corollary 3.2.4 and use only the trivial estimate from the Fourier spectrum coming from concavity, we obtain a restriction theorem in terms of the Sobolev and Fourier dimensions of μ . This result, despite being stated in terms of familiar notions of dimension, has been hidden from view up until now perhaps because it is obtained using the Fourier spectrum for a necessarily intermediate $\theta \in (0, 1)$.

While the L^∞ -dimension gives information regarding the scaling properties of a measure, the Sobolev dimension quantifies how rough the measure is, where roughness is measured in the Sobolev scale. For surface measures on curved manifolds such as the sphere or the cone, these two notions are equivalent. However, for fractal measures the L^∞ -dimension will often be strictly smaller than the Sobolev dimension.

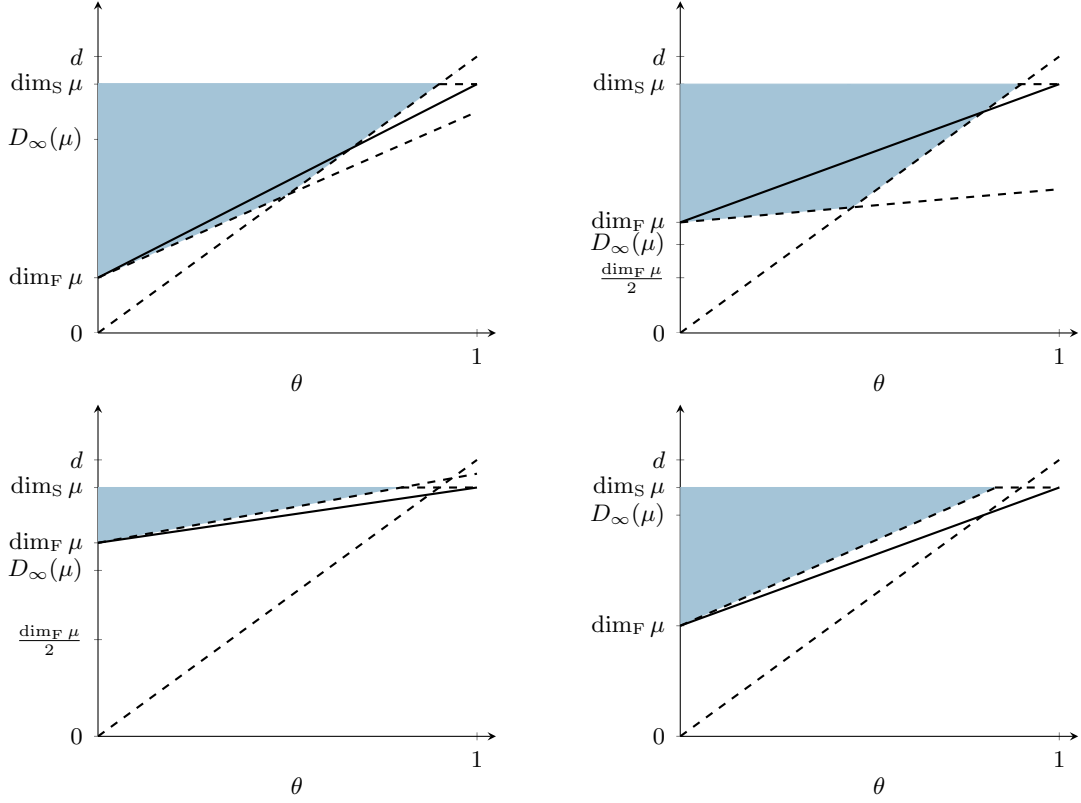


FIGURE 7. In order to improve the Stein–Tomas range for the restriction problem, we need the Fourier spectrum of μ to intersect the shaded region, i.e. for some $\theta \in [0, 1]$ we need the point $(\theta, \dim_F^\theta \mu)$ to lie in the shaded region. Top left: when $\dim_S \mu \geq D_\infty(\mu) + \frac{\dim_F \mu}{2}$ and $D_\infty(\mu) > \dim_F \mu$. Top right: when $\dim_S \mu > D_\infty(\mu) + \frac{\dim_F \mu}{2}$ and $D_\infty(\mu) < \dim_F \mu$. Bottom left: when $\dim_S \mu < D_\infty(\mu) + \frac{\dim_F \mu}{2}$ and $D_\infty(\mu) < \dim_F \mu$. Bottom right: when $\dim_S \mu < D_\infty(\mu) + \frac{\dim_F \mu}{2}$ and $D_\infty(\mu) > \dim_F \mu$. The dashed lines are $\theta(D_\infty(\mu) - \frac{\dim_F \mu}{2}) + \dim_F \mu$ and $d\theta$, and the solid line is $\theta(\dim_S \mu - \dim_F \mu) + \dim_F \mu$, which by concavity is always a lower bound for the Fourier spectrum of μ .

Applying Theorem 3.2.1 with θ at the right end of the interval of allowable θ and using concavity of the Fourier spectrum, we get the following restriction theorem for the Sobolev dimension. In fact, it follows by applying Corollary 3.2.4, which we note because its proof is direct from the Fourier spectrum and does not involve the $L^2 \rightarrow L^2$ estimate coming from the Frostman exponent.

COROLLARY 3.5.1. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$ with $\dim_S \mu < d$. If*

$$q > 4 + \frac{4(d - \dim_S \mu)}{\dim_F \mu},$$

then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, if

$$1 \leq q' < 1 + \frac{\dim_F \mu}{4(d - \dim_S \mu + \dim_F \mu) - \dim_F \mu},$$

then for all $f \in L^{q'}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

PROOF. By concavity of the Fourier spectrum for measures, $\dim_F^\theta \mu \geq \theta(\dim_S \mu - \dim_F \mu) + \dim_F \mu$. Therefore, choosing θ such that $\theta(\dim_S \mu - \dim_F \mu) + \dim_F \mu = d\theta$, i.e. $\theta = \frac{\dim_F \mu}{d - \dim_S \mu + \dim_F \mu}$, Corollary 3.2.4 directly gives that the desired restriction estimate holds for

$$q > \frac{4}{\theta} = \frac{4(d - \dim_S \mu + \dim_F \mu)}{\dim_F \mu},$$

as required. $\square$

Similarly, by concavity (see [37, Proof of Theorem 1.1]), we obtain the following corollary from Theorem 3.2.2.

COROLLARY 3.5.2. *Let $0 < s < d$, $0 < t < s$, and let μ be a non-zero, compactly supported, Borel measure on $\mathbb{R}^d$, such that*

$$I_s(\mu) < \infty;$$

and

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\frac{t}{2}}.$$

Then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^{4+\frac{4(d-s)}{t}}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, for all $f \in L^{1+\frac{t}{4(d-s+t)-t}}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{1+\frac{t}{4(d-s+t)-t}}(\mathbb{R}^d)}.$$

The previous corollaries beat the Stein–Tomas range for measures μ with $D_\infty(\mu) + \dim_F \mu/2 < \dim_S \mu$. For fractal or multifractal measures this is quite typical, since the Fourier dimension is often small and the L^∞ -dimension is often strictly smaller than the Sobolev dimension (the L^∞ -dimension can even be as small as $\dim_F \mu/2$ [77]). On the other hand, if the L^∞ and Sobolev dimensions agree, then the previous corollary never beats Stein–Tomas, and this is the case for surface measures. That said, Corollary 3.2.4 may beat Stein–Tomas, even for surface measures.

3.6. Fourier restriction for the cone

In this section we exhibit our main theorem by applying it to an important example in restriction theory. Let $d \geq 3$ and define the truncated cone in $\mathbb{R}^d$ by

$$C^{d-1} = \{(t, |t|) : t \in \mathbb{R}^{d-1}, |t| \leq 1\} \subseteq \mathbb{R}^d,$$

and let ν_{d-1} be the surface measure on the cone C^{d-1} . This is a well-known and important example in restriction theory.

CONJECTURE 3.6.1 (cone restriction conjecture). *The estimate $\|\widehat{f}\|_{L^{p'}(\nu_{d-1})} \lesssim_{p,q} \|f\|_{L^{q'}(\mathbb{R}^d)}$ holds for all $f \in L^{q'}(\mathbb{R}^d)$ if and only if*

$$\frac{d-2}{p'} \geq \frac{d}{q} \quad \text{and} \quad q > \frac{2(d-1)}{d-2},$$

and for $p = 2$ if and only if

$$q \geq \frac{2d}{d-2}.$$

The general restriction conjecture for the cone has been proven only in dimensions $d = 3, 4, 5$ by Bacelo [3], Wolff [117], and Ou–Wang [95], respectively. The case $p = 2$ was established in unpublished work of Córdoba and Stein in all dimensions, see also [108] where Strichartz generalised the result to quadratic surfaces.

In order to apply our results to the surface measure on the cone, we first derive the Fourier spectrum for this measure. This result may be of independent interest.

PROPOSITION 3.6.2. *Let $d \geq 3$ be an integer, and ν_{d-1} be the surface measure on the cone. Then*

$$\dim_{\mathbb{F}}^{\theta} \nu_{d-1} = \min \left\{ 2 + (d-1)\theta, (d-2) + \theta \right\}$$

for all $\theta \in [0, 1]$. This has a phase transition at $\theta = \frac{d-4}{d-2}$ whenever $d \geq 5$, but for $d = 3, 4$ it is affine. Moreover, the Fourier dimension of ν_{d-1} is $\min\{2, d-2\}$ and the Sobolev (and L^{∞}) dimension of ν_{d-1} is $d-1$.

We defer the proof of the theorem to the end of the section; see Subsection 3.6.1.

From Theorem 3.2.1 we know that $\|\widehat{f\nu_{d-1}}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\nu_{d-1})}$ if

$$q > 2 + 2 \inf_{\substack{\theta \in [0,1] \\ \dim_{\mathbb{F}}^{\theta} \mu > d\theta}} \frac{(d-\alpha)(2-\theta)}{\dim_{\mathbb{F}}^{\theta} \mu - \alpha\theta},$$

For $d = 3, 4$ the spectrum is affine, and we can do nothing more than recover Stein–Tomas (respectively, $q > 6, 4$), but this range is sharp in both of these cases. For $d \geq 5$ we can do better than Stein–Tomas. Here the infimum above is realised at the phase transition $\theta = \frac{d-4}{d-2}$ and, therefore, for any $f \in L^2(\nu_{d-1})$, the estimate

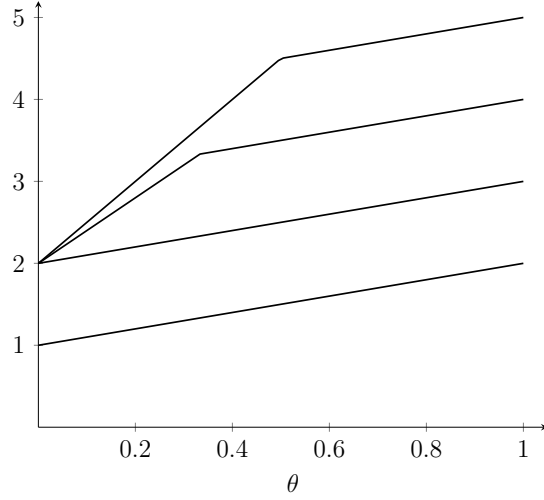


FIGURE 8. The Fourier spectrum of ν_{d-1} on the cone C^{d-1} for $d = 3, \dots, 6$; see Proposition 3.6.2.

$\|\widehat{f\nu_{d-1}}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\nu_{d-1})}$ holds if

$$q > \frac{3d-4}{d-2}.$$

The Stein–Tomas range for all $d \geq 5$ is $q > 4$. We also get a converse result from Theorem 3.2.6. For all $d \geq 3$, the Fourier spectrum crosses the diagonal $d\theta$ at $\theta = \frac{d-2}{d-1}$ and so Theorem 3.2.6 implies that the general restriction estimate (3.1.1) fails for

$$q < \frac{2(d-1)}{d-2}.$$

This outperforms the estimate $q < \frac{2d}{\dim_S \nu_{d-1}} = \frac{2d}{d-1}$ for all $d \geq 3$ and, moreover, recovers the second condition in the cone restriction conjecture 3.6.1.

3.6.1. Proof of Proposition 3.6.2.

Throughout the proof we write $z = (x, y) \in \mathbb{R}^d$ with $x \in \mathbb{R}^{d-1}$ and $y \in \mathbb{R}$ and $(u, v) \in \mathbb{R}^d$ with $u \in \mathbb{R}^{d-1}$ and $v \in \mathbb{R}$. We also write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for real numbers a, b . First observe that, up to normalisation constant, ν_{d-1} disintegrates as

$$\int f(u, v) d\nu_{d-1}(u, v) = \int_0^1 v^{d-2} \int_{S^{d-2}} f(uv, v) d\sigma_{d-2}(u) dv$$

for $f \in L^1(\nu_{d-1})$ where σ_{d-2} is the surface measure on the $(d-2)$ -dimensional sphere S^{d-2} in $\mathbb{R}^{d-1}$. Therefore,

$$\widehat{\nu_{d-1}}(z) = \int_0^1 v^{d-2} \int_{S^{d-2}} e^{-2\pi i(uv \cdot x + v \cdot y)} d\sigma_{d-2}(u) dv$$

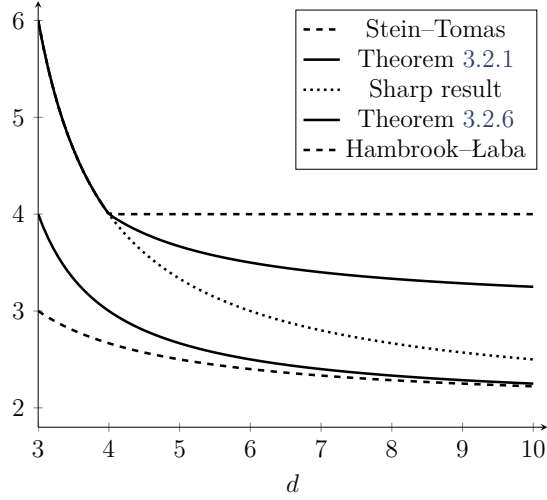


FIGURE 9. Bounds for the range of q for the restriction estimate (3.1.2) to hold for the cone in $\mathbb{R}^d$. The dashed lines are the Stein–Tomas upper bound and the Hambrook–Łaba lower bound, the dotted line is the sharp result, and the solid lines are our upper and lower bounds for the threshold. These plots should be understood as only applying to integer points in the domain, but we included the full curve for aesthetic reasons.

$$\begin{aligned}
&= \int_0^1 v^{d-2} \widehat{\sigma_{d-2}}(vx) e^{-2\pi i y v} dv \\
&= c(d-1) |x|^{\frac{3-d}{2}} \int_0^1 v^{\frac{d-1}{2}} J_{\frac{d-3}{2}}(2\pi v |x|) e^{-2\pi i y v} dv \quad (\text{by [83, (3.41)]})
\end{aligned}$$

where $c(d-1)$ is a constant depending on the ambient dimension $d-1$ and $J_{\frac{d-3}{2}}$ is a Bessel function. We use the above expression to derive estimates for $|\widehat{\nu_{d-1}}(z)|$ in different situations depending on x, y .

Recall the standard asymptotic for Bessel functions (e.g. [83, (3.37)]) which gives

$$(3.6.1) \quad J_m(t) = \frac{\sqrt{2}}{\sqrt{\pi t}} \cos\left(t - \pi m/2 - \pi/4\right) + R_m(t)$$

where R_m is a smooth error function satisfying

$$|R_m(t)| \lesssim t^{-3/2}.$$

Inserting this above gives

$$\begin{aligned}
|\widehat{\nu_{d-1}}(z)| &\lesssim |x|^{\frac{2-d}{2}} \left| \int_0^1 v^{\frac{d-2}{2}} \cos\left(2\pi v |x| - \frac{\pi}{4}(d-2)\right) e^{-2\pi i y v} dv \right| \\
(3.6.2) \quad &+ |x|^{\frac{3-d}{2}} \left| \int_0^1 v^{\frac{d-1}{2}} R_{\frac{d-3}{2}}(2\pi v |x|) e^{-2\pi i y v} dv \right|.
\end{aligned}$$

We estimate these integrals separately, starting with the first one. Both estimates use the method of stationary phase. Using Euler's identity,

$$(3.6.3) \quad 2 \left| \int_0^1 v^{\frac{d-2}{2}} \cos(2\pi v|x| - \frac{\pi}{4}(d-2)) e^{-2\pi i y v} dv \right| \leq \left| \int_0^1 v^{\frac{d-2}{2}} e^{-2\pi i y v + i(2\pi v|x| - \frac{\pi}{4}(d-2))} dv \right| + \left| \int_0^1 v^{\frac{d-2}{2}} e^{-2\pi i y v - i(2\pi v|x| - \frac{\pi}{4}(d-2))} dv \right|.$$

Both of these integrals can be expressed as

$$\int_0^1 \psi(v) e^{i|y|\phi(v)} dv$$

for

$$\psi(v) = v^{\frac{d-2}{2}} e^{\pm i \frac{\pi}{4}(d-2)}$$

and

$$\phi(v) = -2\pi v \left(\operatorname{sgn}(y) \pm \frac{|x|}{|y|} \right).$$

Then

$$\phi'(v) \equiv -2\pi \left(\operatorname{sgn}(y) \pm \frac{|x|}{|y|} \right)$$

and, by the stationary phase estimate given in [83, Corollary 14.3],

$$\left| \int_0^1 \psi(v) e^{i|y|\phi(v)} dv \right| \lesssim |y|^{-1} \left(|\psi(1)| + \int_0^1 |\psi'(v)| dv \right) \lesssim |y|^{-1}$$

whenever $\frac{|x|}{|y|}$ is bounded away from 1. This gives the following bound for the first term in (3.6.2):

$$(3.6.4) \quad |x|^{\frac{2-d}{2}} \left| \int_0^1 v^{\frac{d-2}{2}} \cos(2\pi v|x| - \frac{\pi}{4}(d-2)) e^{-2\pi i y v} dv \right| \lesssim |x|^{\frac{2-d}{2}} |y|^{-1}$$

whenever $\frac{|x|}{|y|}$ is bounded away from 1.

Now we consider the second term from (3.6.2). By the stationary phase estimate given in [83, Corollary 14.3], we get

$$\begin{aligned} \left| \int_0^1 v^{\frac{d-1}{2}} R_{\frac{d-3}{2}}(2\pi v|x|) e^{-2\pi i y v} dv \right| &\lesssim |y|^{-1} \left(|R_{\frac{d-3}{2}}(2\pi|x|)| + \int_0^1 \left| \frac{d}{dv} v^{\frac{d-1}{2}} R_{\frac{d-3}{2}}(2\pi v|x|) \right| dv \right) \\ &\lesssim |y|^{-1} (|x|^{-3/2} + |x|^{-1/2}) \end{aligned}$$

where we have used basic properties of Bessel functions (see [83, (3.36), (3.37), and (3.38)]) to obtain the final estimate. This gives the following bound for the second term in (3.6.2):

$$(3.6.5) \quad |x|^{\frac{3-d}{2}} \left| \int_0^1 v^{\frac{d-1}{2}} R_{\frac{d-3}{2}}(2\pi v|x|) e^{-2\pi i y v} dv \right| \lesssim |x|^{\frac{2-d}{2}} |y|^{-1}.$$

Combining (3.6.4), (3.6.5) and (3.6.2) we get

$$(3.6.6) \quad |\widehat{\nu_{d-1}}(z)| \lesssim |x|^{\frac{2-d}{2}} |y|^{-1}$$

whenever $\frac{|x|}{|y|}$ is bounded away from 1. Note that the above calculation in fact gives

$$(3.6.7) \quad |\widehat{\nu_{d-1}}(z)| \lesssim (|x|^{\frac{2-d}{2}} \wedge 1) (|y|^{-1} \wedge 1),$$

whenever $\frac{|x|}{|y|}$ is bounded away from 1. This aesthetically less pleasing but sharper estimate will be used in some cases below.

For $\frac{|x|}{|y|}$ close to 1 we need a different approach. By (3.6.2), we always have the bound

$$(3.6.8) \quad |\widehat{\nu_{d-1}}(z)| \lesssim |x|^{\frac{2-d}{2}}$$

but can beat this estimate even when $\frac{|x|}{|y|}$ is quite close to 1. For $\frac{|x|}{|y|} \neq 1$ we re-express the integrals from (3.6.3) as

$$\int_0^1 \psi(v) e^{i|y \pm |x||\phi(v)} dv$$

for

$$\psi(v) = v^{\frac{d-2}{2}} e^{\pm i \frac{\pi}{4}(d-2)}$$

and

$$\phi(v) = -2\pi v \operatorname{sgn}(y \pm |x|).$$

Then [83, Corollary 14.3] gives

$$\int_0^1 \psi(v) e^{i|y \pm |x||\phi(v)} dv \lesssim |y \pm |x||^{-1}$$

and, using (3.6.2),

$$(3.6.9) \quad |\widehat{\nu_{d-1}}(z)| \lesssim |x|^{\frac{2-d}{2}} |y \pm |x||^{-1}$$

which beats (3.6.8) whenever $|y \pm |x|| \geq 1$. These estimates are enough for our purposes.

From now on fix $\theta \in (0, 1]$. We begin with the lower bound. Write

$$\begin{aligned} \mathcal{J}_{s,\theta}(\nu_{d-1})^{1/\theta} &= \int_{\substack{z=(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}: \\ \frac{|x|}{|y|} \notin (1/2, 2)}} |\widehat{\nu_{d-1}}(z)|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d} dz \\ &\quad + \int_{\substack{z=(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}: \\ \frac{|x|}{|y|} \in (1/2, 2) \text{ and } |y \pm |x|| \geq 1}} |\widehat{\nu_{d-1}}(z)|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d} dz \\ &\quad + \int_{\substack{z=(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}: \\ \frac{|x|}{|y|} \in (1/2, 2) \text{ and } |y \pm |x|| < 1}} |\widehat{\nu_{d-1}}(z)|^{\frac{2}{\theta}} |z|^{\frac{s}{\theta}-d} dz \end{aligned}$$

$$=: I_1 + I_2 + I_3.$$

We estimate these three integrals separately. Let $B_k(0, r)$ denote the open r -ball in $\mathbb{R}^k$ centred at the origin. First, using (3.6.6) and (3.6.7),

$$\begin{aligned} I_1 &\lesssim \int_{\substack{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}: \\ |x| \geq |y| \geq 1}} \left(|x|^{\frac{2-d}{2}} |y|^{-1} \right)^{\frac{2}{\theta}} |x|^{\frac{s}{\theta}-d} dx dy \\ &\quad + \int_{\substack{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}: \\ 1 \leq |x| \leq |y|}} \left(|x|^{\frac{2-d}{2}} |y|^{-1} \right)^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-d} dx dy \\ &= \int_{x \in \mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{\frac{s}{\theta}-d+\frac{2-d}{\theta}} \int_{\substack{y \in \mathbb{R}: \\ 1 \leq |y| \leq |x|}} |y|^{-\frac{2}{\theta}} dy dx \\ &\quad + \int_{y \in \mathbb{R} \setminus B_1(0,1)} |y|^{\frac{s}{\theta}-d-\frac{2}{\theta}} \int_{\substack{x \in \mathbb{R}^{d-1}: \\ 1 \leq |x| \leq |y|}} |x|^{\frac{2-d}{\theta}} dx dy \\ &\lesssim \int_{x \in \mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{\frac{s}{\theta}-d+\frac{2-d}{\theta}} dx + \int_{y \in \mathbb{R} \setminus B_1(0,1)} |y|^{\frac{s}{\theta}-d-\frac{2}{\theta}+(d-1+\frac{2-d}{\theta}) \vee 0} dy \\ &< \infty \end{aligned}$$

provided

$$\frac{s}{\theta} - d + \frac{2-d}{\theta} < 1 - d$$

and

$$\frac{s}{\theta} - d - \frac{2}{\theta} + \left(d - 1 + \frac{2-d}{\theta} \right) \vee 0 < -1,$$

that is, provided

$$s < \min \left\{ 2 + (d-1)\theta, (d-2) + \theta \right\}$$

which is consistent with the desired lower bound.

Turning to the second integral, using (3.6.9)

$$\begin{aligned} I_2 &\lesssim \int_{\substack{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}: \\ \frac{|x|}{|y|} \in (1/2, 2) \text{ and } |y \pm |x|| \geq 1}} \left(|x|^{\frac{2-d}{2}} |y \pm |x||^{-1} \right)^{\frac{2}{\theta}} |x|^{\frac{s}{\theta}-d} dx dy \\ &\lesssim \int_{\mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{\frac{2+s-d}{\theta}-d} \int_{\substack{y \in \mathbb{R}: \\ \frac{|x|}{|y|} \in (1/2, 2) \text{ and } |y \pm |x|| \geq 1}} |y \pm |x||^{-\frac{2}{\theta}} dy dx \\ &\lesssim \int_{\mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{\frac{2+s-d}{\theta}-d} \int_1^{|x|} \alpha^{-\frac{2}{\theta}} d\alpha dx \\ &\lesssim \int_{\mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{\frac{2+s-d}{\theta}-d} dx \\ &< \infty \end{aligned}$$

provided

$$\frac{2+s-d}{\theta} - d < 1-d,$$

that is, provided

$$s < (d-2) + \theta$$

which is consistent with the desired lower bound.

Finally, for the third integral, using (3.6.8),

$$\begin{aligned} I_3 &\lesssim \int_{x \in \mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} \int_{\substack{y \in \mathbb{R}: \\ |y \pm x| < 1}} |x|^{\frac{2-d}{\theta}} |x|^{\frac{s}{\theta}-d} dy dx \\ &\lesssim \int_{x \in \mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{\frac{2+s-d}{\theta}-d} dx \\ &< \infty \end{aligned}$$

provided

$$\frac{2+s-d}{\theta} - d < 1-d,$$

that is, provided

$$s < (d-2) + \theta$$

which is consistent with the desired lower bound. Combining our estimates for I_1 , I_2 and I_3 , gives the desired lower bound.

We turn our attention to the upper bound. Fix $\varepsilon \in (0, 0.01)$ small. Then

$$\begin{aligned} \mathcal{J}_{s,\theta}(\nu_{d-1})^{1/\theta} &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\widehat{\nu_{d-1}}(x, y)|^{\frac{2}{\theta}} (|x| \vee |y|)^{\frac{s}{\theta}-d} dy dx \\ &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left| \int_0^1 v^{d-2} \widehat{\sigma_{d-2}}(xv) e^{-2\pi i y \cdot v} dv \right|^{\frac{2}{\theta}} (|x| \vee |y|)^{\frac{s}{\theta}-d} dy dx \\ &\geq \int_{\mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} \int_{B_1(0,\varepsilon)} \left| \int_0^1 v^{d-2} \widehat{\sigma_{d-2}}(xv) e^{-2\pi i y \cdot v} dv \right|^{\frac{2}{\theta}} |x|^{\frac{s}{\theta}-d} dy dx \\ &\gtrsim_{\varepsilon} \int_{\mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} \left| \int_0^1 v^{d-2} \widehat{\sigma_{d-2}}(xv) dv \right|^{\frac{2}{\theta}} |x|^{\frac{s}{\theta}-d} dx \\ &\approx \int_{\mathbb{R}^{d-1} \setminus B_{d-1}(0,1)} |x|^{-\frac{d-2}{\theta}} |x|^{\frac{s}{\theta}-d} dx \\ &= \infty \end{aligned}$$

whenever $-\frac{d-2}{\theta} + \frac{s}{\theta} - d \geq -(d-1)$, proving $\dim_{\mathbb{F}}^{\theta} \nu_{d-1} \leq (d-2) + \theta$. Similarly,

$$\mathcal{J}_{s,\theta}(\nu_{d-1})^{1/\theta} \geq \int_{\mathbb{R} \setminus B_1(0,1)} \int_{B_{d-1}(0,\varepsilon)} \left| \int_0^1 v^{d-2} \widehat{\sigma_{d-2}}(xv) e^{-2\pi i y \cdot v} dv \right|^{\frac{2}{\theta}} |y|^{\frac{s}{\theta}-d} dx dy$$

$$\begin{aligned}
&\gtrsim_{\varepsilon} \int_{\mathbb{R} \setminus B_1(0,1)} |y|^{-\frac{2}{\theta}} |y|^{\frac{s}{\theta}-d} dy \\
&= \infty
\end{aligned}$$

whenever $-\frac{2}{\theta} + \frac{s}{\theta} - d \geq -1$, proving $\dim_{\mathbb{F}}^{\theta} \nu_{d-1} \leq 2 + (d-1)\theta$. This completes the proof.

3.7. Fourier restriction for the moment curve

In this section we consider another important example in restriction theory and consider how our results compare with the Stein–Tomas range. For $d \geq 2$, let the moment curve (or Veronese curve) be defined by

$$V^d \{ (t, t^2/2, \dots, t^d/d!) : t \in [0, 1] \} \subseteq \mathbb{R}^d$$

and let ν denote the pushforward of Lebesgue measure onto the curve. Thus, ν is—up to a bi-Lipschitz density—the arclength measure on V^d . Despite being a 1-dimensional curve embedded in high ambient dimension, one can still get good estimates for Fourier decay and restriction due to the curvature. It is known that the extension estimate

$$(3.7.1) \quad \|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mu)}$$

holds if and only if

$$(3.7.2) \quad q \geq \frac{d(d+1)p'}{2} \quad \text{and} \quad q > \frac{d^2 + d + 2}{2},$$

and for $p = 2$ if and only if $q \geq d^2 + d$; see [25]. We are most grateful to Jonathan Hickman for both suggesting the moment curve as an example to consider and completely solving it for us by pointing out that an explicit formula for the Fourier spectrum follows from estimates in [11] for the averages

$$G_{\rho}(R) := \left(\int_{S^{d-1}} |\widehat{\nu}(R\omega)|^{\rho} d\sigma_{d-1}(\omega) \right)^{1/\rho}.$$

In particular, setting $\rho = \frac{2}{\theta}$ and using polar coordinates,

$$\mathcal{J}_{s,\theta}(\nu)^{1/\theta} = \int_0^{\infty} R^{\frac{s}{\theta}-1} G_{\frac{2}{\theta}}(R)^{\frac{2}{\theta}} dR$$

and applying [11, Theorems 1.2–1.3, for $K = d$], we get the following. This example is noteworthy because we have not yet seen a natural example exhibiting multiple phase transitions.

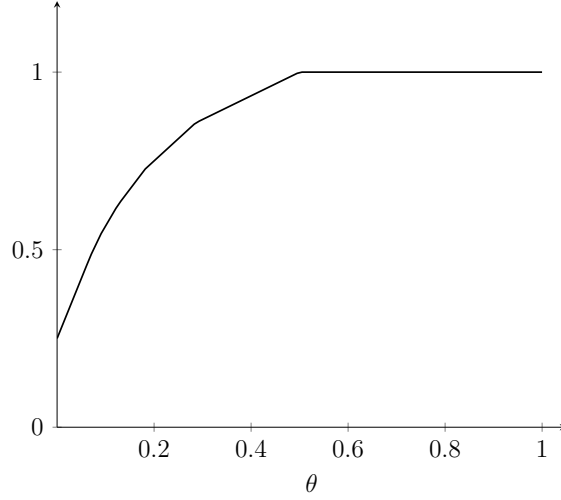


FIGURE 10. The Fourier spectrum of the arclength measure on the moment curve in $\mathbb{R}^8$; see Proposition 3.7.1. There are 6 phase transitions and the Fourier dimension is $1/4$.

PROPOSITION 3.7.1. *Let $d \geq 2$ be an integer and ν be the arclength measure on the moment curve. Then*

$$\dim_{\mathbb{F}}^{\theta} \nu = \min_{k=2, \dots, d} \frac{2}{k} + \frac{k^2 - k - 2}{2k} \theta.$$

In particular, $\dim_{\mathbb{F}} \nu = 2/d$, the Fourier spectrum coincides with the Hausdorff dimension of the curve for $\theta \geq 1/2$ and the Fourier spectrum has $d - 2$ phase transitions occurring at

$$\theta = \frac{4}{k^2 - k + 2} \quad (k = 3, \dots, d).$$

Finally, $\dim_{\mathbb{F}}^{\theta} \nu > d\theta$ holds for all

$$0 \leq \theta < \frac{4}{d^2 + d + 2}.$$

In fact [11] treats much more general curves, in particular, allowing less curvature than the moment curve. By carefully examining their results, one can derive the Fourier spectrum more generally, but we leave this to the interested reader.

Let us now examine extension estimates coming from Proposition 3.7.1. For $p = 2$, the Stein–Tomas range is

$$q \geq 2d^2 - 2d + 2$$

and Theorem 3.2.1 obtains the range

$$q > d^2 + d + 2,$$

which is the sharp bound plus 2. Moreover, for $d \geq 4$ the range in Theorem 3.2.1 is optimised (uniquely) at $\theta = \frac{4}{d^2 + d + 2}$. On the other hand, Theorem 3.2.6 shows that the

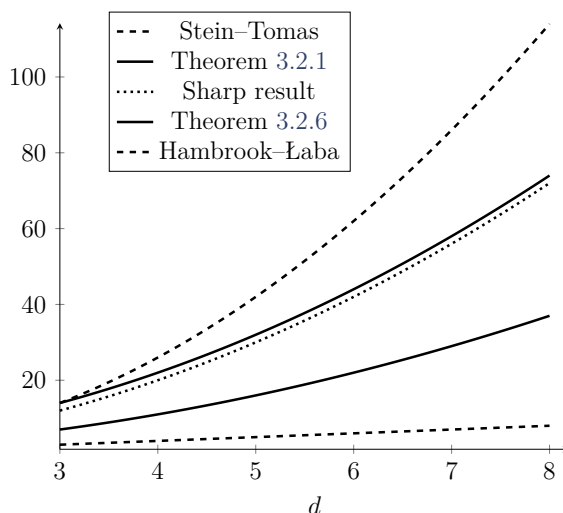


FIGURE 11. Bounds for the range of q for the extension estimate (3.1.2) to hold for the moment curve in $\mathbb{R}^d$. The dashed lines are the Stein–Tomas upper bound and the Hambrook–Łaba lower bound, the dotted line is the sharp result, and the solid lines are our upper and lower bounds for the threshold. These plots should be understood as only applying to integer points in the domain, but we included the full curve for aesthetic reasons.

general extension estimate (3.7.1) fails for

$$q < \frac{d^2 + d + 2}{2},$$

whereas applying the Hambrook–Łaba bound only gives failure for $q < d$. It is perhaps interesting to note that the bound coming from Theorem 3.2.6 is sharp and provides the right-hand side estimate in (3.7.2). This was also the case for the cone, recall Conjecture 3.6.1, and the sphere, recall the discussion after Theorem 3.2.6.

3.8. Fourier restriction on fractals

3.8.1. Multifractal measures on the 1/3-Cantor set.

We saw in Lemma 3.2.3 that it does not make sense to restrict the Fourier transform to measures that do not have Fourier decay. That said, examples of multifractal measures with positive Fourier dimension abound. Indeed, in [105] Solomyak proved that outside of a set of parameters of Hausdorff dimension zero, all self-similar measures supported on the line have positive Fourier dimension. However, finding explicit examples of such measures can be very difficult, and since we do not know of any, we will add a small amount of ‘noise’ to a concrete family of multifractal measures, which will result in a new family that will exhibit both multifractal behaviour and polynomial Fourier decay.

Let $p > 1/2$ and μ_p be the (self-similar) measure on the $1/3$ Cantor set associated to probabilities p and $(1-p)$. Then, by (1.4.1),

$$\dim_F \mu_p = 0; \quad D_\infty(\mu)_p = \frac{\log p}{-\log 3}; \quad \dim_S \mu_p = \frac{\log(p^2 + (1-p)^2)}{-\log 3}.$$

Let $\varepsilon > 0$ and ν_ε be a Salem measure of dimension ε .

Consider the measure $m = \mu_p * \nu_\varepsilon$. By [37, Theorem 6.1], for all $\theta \in [0, 1]$, $\dim_F^\theta m \geq \dim_F^\theta \mu_p + \varepsilon$.

If $\dim_F^{1/2} m > 1/2$, we may fix $\theta = 1/2$ in Theorem 3.2.1, which gives the range

$$(3.8.1) \quad q > 2 + \frac{2(1-\alpha)(2-\frac{1}{2})}{\dim_F^{1/2} m - \frac{\alpha}{2}} = 2 + \frac{6(1-\alpha)}{2 \dim_F^{1/2} m - \alpha}$$

for the restriction estimate to hold, where $\alpha = D_\infty(m)$.

Since $\mu_p * \mu_p$ is a self-similar measure satisfying the open set condition with a system of 3 maps and weights p^2 , $(1-p)^2$ and $2p(1-p)$, then the Fourier spectrum of μ_p at $\theta = 1/2$ is, by Lemma 1.5.1,

$$\dim_F^{1/2} \mu_p = \frac{\dim_S(\mu_p * \mu_p)}{2} = \frac{\log(p^4 + (2p(1-p))^2 + (1-p)^4)}{-2 \log 3}.$$

Therefore,

$$\dim_F^{1/2} m \geq \frac{\log(p^4 + (2p(1-p))^2 + (1-p)^4)}{-2 \log 3} + \varepsilon,$$

where we choose ε small, but large enough so that $\dim_F^{1/2} m > 1/2$.

To bound $D_\infty(m)$ from below let $\beta < D_\infty(\mu)$, $\delta < \varepsilon = D_\infty(\nu_\varepsilon)$, $x \in \mathbb{R}$ and $r > 0$. Then

$$\begin{aligned} m(B(x, r)) &= \iint 1_{B(x, r)}(u + v) d\mu(u) d\nu_\varepsilon(v) \\ &= \int_{B(x, r)} \int_{B(x, r)-v} d\mu(u) d\nu_\varepsilon(v) \\ &= \int_{B(x, r)} \mu(B(x - v, r)) d\nu_\varepsilon(v) \\ &\lesssim r^{\beta+\delta}, \end{aligned}$$

and letting $\beta \rightarrow D_\infty(\mu)$ and $\delta \rightarrow \varepsilon$ shows that $D_\infty(m) \geq D_\infty(\mu_p) + \varepsilon$.

Since $\dim_F^{1/2} m > 1/2$, as a function of $D_\infty(m)$, $\frac{6(1-D_\infty(m))}{2 \dim_F^{1/2} m - D_\infty(m)}$ is decreasing. Replacing the obtained bounds on the right-hand side of (3.8.1) we get that restriction is possible if

$$(3.8.2) \quad q > 2 + \frac{6(\log 3 + \log p - \varepsilon \log 3)}{\log p + \varepsilon \log 3 - \log(p^4 + (2p(1-p))^2 + (1-p)^4)}.$$

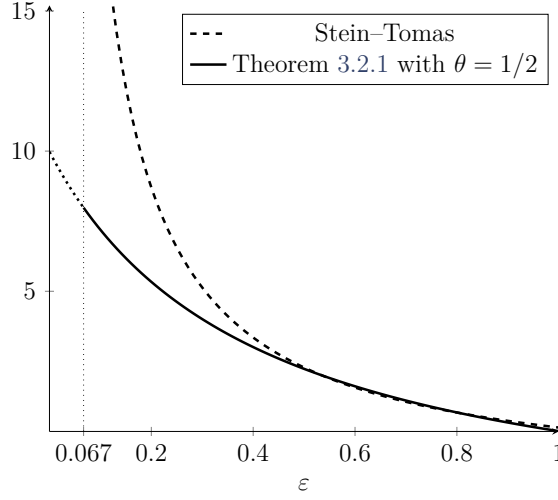


FIGURE 12. Lower bound on the range of q for the restriction estimate (3.1.2) to hold for the measure $\mu_{0.6} * \nu_\varepsilon$, as a function of ε . Note that the bound obtained from Theorem 3.2.1 is only valid for values of ε greater than 0.067.

On the other hand, Stein–Tomas gives the range

$$(3.8.3) \quad q > 2 + \frac{4(\log 3 + \log p - \varepsilon \log 3)}{\varepsilon \log 3}.$$

For example, if $p = 0.6$, to have $\dim_{\mathbb{F}}^{1/2} m > 1/2$ we need $\varepsilon \geq 0.067$. Our results would still apply for smaller ε (and always give a non-trivial range for restriction, see Section 3.8.2) but we would need to use a smaller value of θ . The problem is then that we do not know the Fourier spectrum of μ_p explicitly other than at $\theta = 1/2$. For this threshold value of $\varepsilon = 0.067$, the range of (3.8.2) is $q > 7.99$ and the Stein–Tomas range (3.8.3) is $q > 29.95$.

Curiously, if in the previous example we set $\varepsilon = 0.7$, then from Figure 12 or by direct calculation, we can see that the Stein–Tomas range (3.8.3) is better than the one given by Theorem 3.2.1 with $\theta = 1/2$, (3.8.2). This implies that for such values of ε , the improvement on the Stein–Tomas result will come from another value $\theta < 1/2$; see previous discussion and subsequent section.

3.8.2. Measures on a large family of fractals.

Here we consider a variant on the family of examples from the previous section where we replace the concrete family of self-similar measures μ_p with a large family of measures satisfying a mild non-concentration condition (certainly the measures μ_p are in this family as well as many other examples). As in the previous example, these measures do not have polynomial Fourier decay, and so in order to make sense of restriction we will convolve them with a Salem measure of small dimension. If we

make this ‘noise’ small, we get better restriction estimates than those provided by Stein–Tomas.

Let μ be a non-negative, finite, compactly supported, Borel measure on $\mathbb{R}^d$ which satisfies: $\dim_F \mu = 0$, $\dim_S \mu < d$, and that the upper right semi-derivative of $\theta \mapsto \dim_F^\theta \mu$ at $\theta = 0$ is d . In particular, since $\dim_F^\theta \mu \leq \dim_F \mu + d\theta$ this derivative condition says that the Fourier spectrum has the largest possible growth at $\theta = 0$. This seemingly artificial condition turns out to be satisfied very generally due to work of Khalil [67]. We say that a Borel measure μ on $\mathbb{R}^d$ is uniformly affinely non-concentrated if there exist $s > 0$, $C \geq 1$ such that for every $\delta > 0$, $x \in \mathbb{R}^d$, $0 < r \leq 1$ and hyperplane $W \subseteq \mathbb{R}^d$,

$$\mu(W^{(\delta r)} \cap B(x, r)) \leq C\delta^s \mu(B(x, r))$$

where $W^{(\delta r)}$ is the δr neighbourhood of W . In particular, many families of dynamically defined measures satisfy this condition. It turns out that if such a measure μ has Fourier dimension zero, then it satisfies our derivative condition. Indeed, by [37, Lemma 6.2] and [67, Theorem 1.6],

$$\lim_{\theta \rightarrow 0} \frac{\dim_F^\theta \mu'}{\theta} = d.$$

Let $\varepsilon > 0$ be small and ν_ε be a Salem measure on $\mathbb{R}^d$ of dimension ε . We consider the measure $m = \mu * \nu_\varepsilon$ and our goal is to establish a restriction estimate for m with an optimal range as $\varepsilon \rightarrow 0$. Since we assume $\dim_F \mu = 0$, we cannot do any better in general than the estimate $\dim_F m \geq \varepsilon$ and therefore the Stein–Tomas range for restriction (3.1.7) to hold is

$$q > 2 + \frac{4(d - D_\infty(m))}{\varepsilon} = \Omega(\varepsilon^{-1}).$$

Similarly, considering the Fourier spectrum, the best estimate we can get in general is $\dim_F^\theta m \geq \dim_F^\theta \mu + \varepsilon$. Let $\theta_0 \in (0, 1)$ be chosen such that $\dim_F^{\theta_0} \mu + \varepsilon = d\theta_0$, noting that this choice is unique. Then Corollary 3.2.4 gives that restriction (3.1.7) holds for

$$q > \frac{4}{\theta_0} = o(\varepsilon^{-1}).$$

In particular, we obtain an asymptotically better range for restriction as $\varepsilon \rightarrow 0$. The fact that $\theta_0^{-1} = o(\varepsilon^{-1})$ follows since the upper right semi-derivative of $\dim_F^\theta \mu$ at $\theta = 0$ is d . That is, for every $\eta > 0$, there will exist $\theta > 0$ such that $(d - \eta)\theta < \dim_F^\theta \mu$ and thus $\theta_0 > \varepsilon/\eta$ for ε small enough.

In certain situations we can say more. For example, if $\dim_F^\theta \mu$ is twice (right) continuously differentiable at $\theta = 0$ (and still assuming that the first right derivative

is d) then, by Taylor's theorem,

$$\dim_{\mathbb{F}}^{\theta} \mu = d\theta - |O(\theta^2)|$$

and so Corollary 3.2.4 gives that restriction (3.1.7) holds for

$$q > \frac{4}{\theta_0} = O(\varepsilon^{-1/2}).$$

CHAPTER 4

Fourier restriction to the Brownian graph

4.1. Introduction

As previously discussed in Chapter 3, although the Fourier restriction problem is usually cast in the context of smooth manifolds, there is a significant amount of work done for the case where the set is of fractal dimension. In the fractal setting, it was first studied in [85], and saw continuous development in the subsequent years [84, 1, 18].

One of the main aims of this Chapter is to motivate the study of the restriction problem for a natural family of (topological) co-dimension 1 manifolds, which are not smooth, but rather are realised as the graph of Brownian motion or more generally the Brownian sheet. One may consider the graphs of Brownian motion or the Brownian sheet to be the canonical examples of a fractal surface. In some sense, these examples sit in between the fractal and smooth settings and exhibit many interesting geometric and analytical features, making them good candidates for Fourier restriction and the related problem of deriving the Fourier spectrum. Indeed, such graphs are currently attracting a lot of attention in the context of Fourier dimension, see [35, 73, 74].

4.1.1. Fourier restriction and fractal surfaces.

Brownian motion is one of the most important, widely studied, and well understood stochastic processes. It is defined as the unique (random) map $W : [0, 1] \rightarrow \mathbb{R}$ with independent increments, that is almost surely continuous, satisfying $W(0) = 0$ almost surely, and such that for $t, s \in \mathbb{R}$ $t > s$, $W(t) - W(s) \sim N(0, t - s)$. Its fractal properties have been studied since the work of Levy [75] and Taylor [111], and its Fourier analytic properties since Kahane [60, 61].

One may generalise Brownian motion to a map $W : [0, 1]^k \rightarrow \mathbb{R}^l$, which is often referred to as the Brownian sheet or the (k, l) -Brownian sheet. Here, we focus on the case $l = 1$, and we will work with a simpler version of the $(k, 1)$ -Brownian sheet known as the additive Brownian sheet. This reduction makes the process easier to study, while still resembling the standard $(k, 1)$ -Brownian sheet (see [68] for a more detailed discussion on additive processes). We define the $(k, 1)$ -additive Brownian sheet as the

map $W : [0, 1]^k \rightarrow \mathbb{R}$ satisfying that for $t \in [0, 1]^k$, $t = (t_1, \dots, t_k)$,

$$W(t) = \sum_{i=1}^k W^i(t_i),$$

where $W^i : [0, 1] \rightarrow \mathbb{R}$ are independent 1-parameter Brownian motions. This process is usually considered as a good proxy for the non-additive case and many properties which hold in the additive case can be shown to hold also in the non-additive case.

Our main object of study is the graph of the $(k, 1)$ -additive Brownian sheet defined as

$$G(W) = \{(t, W(t)) : t \in [0, 1]^k\},$$

and we will refer to $G(W)$ as the *additive Brownian surface* (or just *Brownian surface* in the non-additive case). The additive Brownian surface is a simply connected manifold of topological co-dimension 1 living in $\mathbb{R}^{k+1}$. It comes with a natural ‘surface measure’, which is simply the lift of Lebesgue measure on $[0, 1]^k$ to the graph. We define the *surface measure* μ on the additive Brownian sheet for Borel $E \subset \mathbb{R}^{k+1}$ by

$$\mu(E) = \mathcal{L}^k(\{t \in [0, 1]^k : (t, W(t)) \in E\}),$$

where $\mathcal{L}^k$ is the k -dimensional Lebesgue measure. Our main question is now the following.

QUESTION 4.1.1. *For which values of p, q does the extension estimate*

$$(4.1.1) \quad \|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mu)}$$

hold almost surely for the surface measure on the Brownian sheet?

We make partial progress towards answering this question in this paper. One may also consider the following less precise question.

QUESTION 4.1.2. *For which values of p, q does the extension estimate (4.1.1) hold almost surely for natural measures supported fractal manifolds generated by important stochastic processes, such as the (non-additive) (k, l) -Brownian sheet, analogous fractional Brownian sheets, Levy processes, the Gaussian free field, etc?*

Although these questions are of independent interest, it might be interesting to further investigate the connection to (stochastic) PDEs. Following the same steps as for the Schrödinger equation (3.1.4), we can define the random operator T to be the Fourier multiplier with symbol $W(\xi)$. That is, T is the operator such that $\widehat{Tu}(\xi, t) = W(\xi)\widehat{u}(\xi, t)$, or $Tu(x, t) = (\widetilde{W} * u)(x, t)$, where $*$ is the convolution operator in the x

variable. Then $u(x, t) = \widehat{f\mu}(x, t)$ is the solution to the equation

$$(4.1.2) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) = 2\pi i T u(x, t), & (x, t) \in [0, 1]^k \times \mathbb{R}; \\ u(x, 0) = f(x), & x \in [0, 1]^k, \end{cases}$$

with $f = \check{g}$ and μ the surface measure on the additive Brownian surface. The Fourier restriction estimate we obtain in Theorem 4.2.4 below implies that whenever $\widehat{f} \in L^2(\mu)$, almost surely the solution of (4.1.2) is $u \in L^q(\mathbb{R}^d)$ for any $q > 4$ if $k = 1$, or $q > \frac{8k+2}{3k}$ if $k \geq 2$. It would be interesting to know if there are useful physical interpretations of these equations or if there are connections to existing PDEs in the literature, but we do not pursue this here.

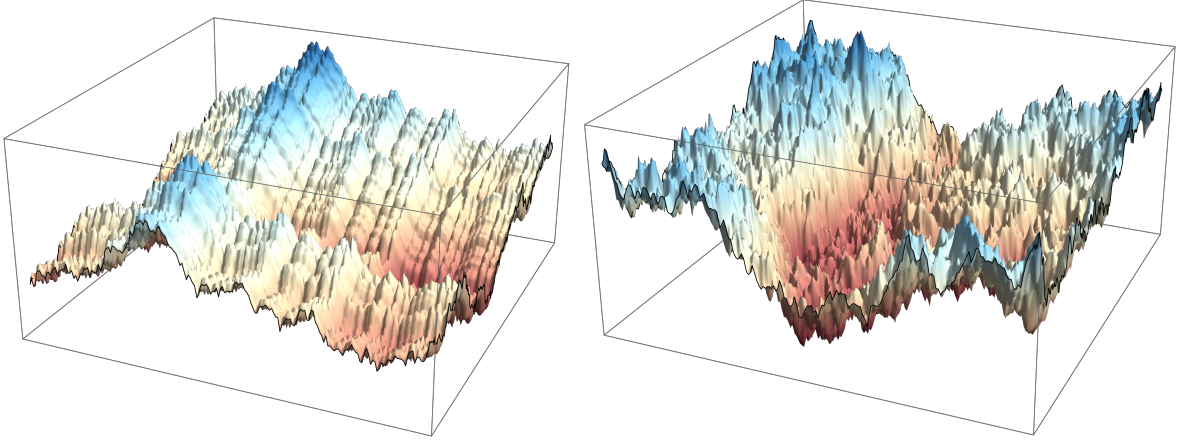


FIGURE 13. Two realisations of the Brownian surface for $k = 2$. Left: in the additive case. Right: in the non-additive case.

4.1.2. Dimension theory for Brownian surfaces and Stein–Tomas.

The dimension theory of the graph of standard Brownian motion was first studied in [111], where it was proved that $\dim_{\text{H}} G(W) = \frac{3}{2}$ almost surely. This result was generalised by [119] to the (non-additive) Brownian sheet, showing that $\dim_{\text{H}} G(W) = k + 1/2$; this formula also holds in the additive case. The Fourier dimension of the graph of standard Brownian motion was shown to be equal to 1 in [42, 35]. The dimensional results mentioned here also hold with the graph replaced by the surface measure, that is, the dimension of the surface is *witnessed* by the surface measure. We show below that this also holds for the Fourier dimension of the additive Brownian surface when $k = 2$ but for $k \geq 3$ one should not expect this to hold. Indeed, one might conjecture that the Fourier dimension of the surface is k , but we show that the Fourier dimension of the surface measure is 2 for all $k \geq 2$.

One reason why these Brownian graphs and surfaces are interesting examples in Fourier restriction theory is that they are *not* Salem sets (they have distinct Hausdorff

and Fourier dimensions). This was perhaps a surprise since Kahane [62] established that many other stochastically defined sets are Salem, including Brownian images. Indeed, it is often expected for random measures to exhibit Fourier decay (see Section 1.5.1).

Recall that the Stein–Tomas theorem gives sufficient conditions for the extension estimate (4.1.1) to hold in the case $p = 2$ which depend on the Fourier and Frostman dimensions of the measure μ . It states that the estimate

$$(4.1.3) \quad \|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$$

holds for all $f \in L^2(\mu)$ as long as $q > 2 + 4 \frac{d - D_\infty(\mu)}{\dim_F \mu}$ (see Section 1.3). As such, we can already get some information for the surface measure on the graph of standard Brownian motion given that the Fourier dimension is 1 and the Frostman dimension is $3/2$. We obtain that the estimate (4.1.3) holds for all $f \in L^2(\mu)$ whenever $q > 4$.

The strategy in this Chapter is to derive a precise almost sure formula for the Fourier spectrum of the surface measure on the additive Brownian surface (see Theorem 4.2.1) and then to use this information in Theorem 3.2.1 to derive non-trivial Fourier restriction estimates (see Theorem 4.2.4). We also consider necessary conditions for Fourier restriction. First we obtain them directly from the Fourier spectrum (see Corollary 4.2.5) using Theorem 3.2.6 and second by constructing a Knapp example (see Theorem 4.2.7 and Theorem 4.2.6) which also works for non-additive Brownian and fractional Brownian surfaces. Curiously, in the case $p = 2$ we obtain the same sufficient condition from both approaches.

4.2. Main Results

Our first main result gives a precise almost sure formula for the Fourier spectrum of the surface measure on the additive Brownian surface. The $\theta = 0$ case gives the Fourier dimension and this result is new for $k \geq 2$. For $k = 1$ it was proved in [35].

THEOREM 4.2.1. *Let μ be the surface measure on the additive Brownian surface. Then, almost surely, for all $\theta \in [0, 1]$,*

$$\dim_F^\theta \mu = \min \left\{ k + \frac{\theta}{2}, 2 + k\theta \right\}.$$

In particular, there is a phase transition at $\theta = \frac{k-2}{k-1/2}$, whenever $k > 2$, and for $k = 1, 2$ it is affine.

We will prove Theorem 4.2.1 in Section 4.3. The previous theorem gives a lower bound for the Fourier spectrum of the Brownian surface itself which is sharp at $\theta = 1$ and is sharp at $\theta = 0$ for $k = 1, 2$. For $k \geq 3$ the lower bound is not sharp and all that

can be said is the general upper bound

$$\dim_{\mathbb{F}}^{\theta} G(W) \leq \min\{k + (k+1)\theta, k + 1/2\}$$

which follows by combining (the higher dimensional version of) the main result from [42] that any graph over $[0, 1]^k$ has Fourier dimension at most k (see [83, Theorem 6.10]), the fact that $\dim_{\mathbb{H}} G(W) = k + 1/2$, and the general upper bound for the Fourier spectrum from Theorem 1.6.1. To obtain the sharp result one would need to consider a different measure, most likely the lift of a Schwartz density to the graph. We record the $\theta = 0$ and $k = 2$ case in the following corollary since it may be interesting in its own right.

COROLLARY 4.2.2. *The Fourier dimension of the additive Brownian surface $G(W)$ in the case $k = 2$ is $\dim_{\mathbb{F}} G(W) = 2$.*

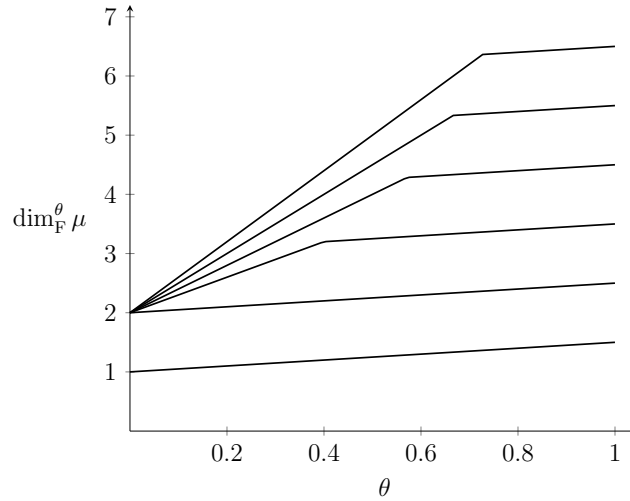


FIGURE 14. Plots of the Fourier spectrum of the surface measure μ on the additive Brownian surface $G(W)$ for $k = 1, \dots, 6$; see Theorem 4.2.1.

We also note the following result regarding the L^∞ -dimension of the surface measure on the additive Brownian surface. This result will be needed to obtain a Fourier restriction estimate (4.2.1) for the additive Brownian surface measure from the Stein–Tomas theorem and its generalisation in Theorem 3.2.1.

PROPOSITION 4.2.3 (Xiao). *Let μ be the surface measure on the additive Brownian surface. Then, almost surely*

$$\dim_{\mathbb{F}} \mu = k + \frac{1}{2}.$$

PROOF. First note that [118, Propostion 3.2] still holds for the additive Brownian surface. Thus, by [118, (3.11)] with $\alpha = 1/2$, there exists $C > 0$ such that for any

$t \in [0, 1]^k$, almost surely

$$\limsup_{r \rightarrow 0} \frac{\mu(B((t, W(t)), r))}{r^{k+\frac{1}{2}} (\log \log 1/r)^{\frac{1}{2k}}} \leq C,$$

which gives $\dim_{\text{Fr}} \mu \geq k + \frac{1}{2}$. This together with the fact that $\dim_{\text{Fr}} \mu \leq \dim_{\text{S}} \mu$ and Theorem 4.2.1 proves the result. $\square$

Next we combine Theorem 4.2.1 with Theorem 3.2.1 to obtain the following Fourier restriction estimate for the surface measure on the additive Brownian surface.

THEOREM 4.2.4. *Let μ be the surface measure on the additive Brownian surface. If $q > 4$ for $k = 1$, or*

$$q > \frac{8k+2}{3k}$$

for $k \geq 2$, then almost surely, for all $f \in L^2(\mu)$,

$$(4.2.1) \quad \|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, if $1 < q < 4/3$ for $k = 1$, or

$$1 \leq q' < \frac{8k+2}{5k+2},$$

for $k \geq 2$, then, almost surely, for all $f \in L^{q'}(\mathbb{R}^d)$,

$$(4.2.2) \quad \|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

PROOF. From Theorem 3.2.1 and Proposition 4.2.3 we get that for all $f \in L^2(\mu)$, (4.2.1) holds whenever

$$q > 2 + \frac{2 - \theta}{\dim_{\text{F}}^{\theta} \mu - (k + 1/2)\theta}$$

for all $\theta \in [0, 1]$. We now optimise this estimate by varying θ and find that the optimal estimate is obtained (uniquely) at the phase transition $\theta = \frac{k-2}{k-1/2}$ for $k > 2$ and (uniquely) at $\theta = 0$ for $k = 1, 2$. $\square$

Note that for $k = 1, 2$, the Fourier extension estimate we obtain is the same as that coming from the Stein–Tomas theorem, that is, $q > 4, 3$ respectively. However, we obtain a better estimate than Stein–Tomas for all $k \geq 3$.

Next we consider the reverse problem of finding necessary conditions for Fourier restriction. First, we obtain the following by combining Theorem 4.2.1 with Theorem 3.2.6.

COROLLARY 4.2.5. *Let μ be the surface measure on the additive Brownian surface. Then, almost surely, (4.1.1) cannot hold whenever*

$$q < 2 + \frac{1}{k}.$$

Equivalently, (4.2.2) cannot hold whenever

$$q' > \frac{2k+1}{k+1}.$$

PROOF. From Theorem 3.2.6, we get that (4.1.1) cannot hold whenever

$$q < \sup\{2/\theta : \dim_{\mathbb{F}}^{\theta} \mu < (k+1)\theta\},$$

and combining this with Theorem 4.2.1 yields the desired estimate. $\square$

Next we derive a general necessary condition for (4.1.1) to hold by building a Knapp example. In fact, all we use to do this is that the additive Brownian sheet is almost surely locally α -Hölder for any $\alpha < 1/2$. This follows easily from the well-known result in the case $k = 1$. Thus, more generally we have the following result which also holds for the non-additive Brownian sheet (with $\alpha < 1/2$) and the analogous fractional Brownian surface with $\alpha < H$ where $H \in (0, 1)$ is the Hurst parameter, as well as many other stochastically defined surfaces.

THEOREM 4.2.6. *Let $f : [0, 1]^k \rightarrow \mathbb{R}$ be such that for some $x \in [0, 1]^k$, f is locally α -Hölder for some $\alpha > 0$ at x . Let μ be the lift of Lebesgue measure onto the graph of f . Then (4.1.1) cannot hold whenever*

$$p < \frac{kq}{k(q-1) - \alpha}.$$

We prove Theorem 4.2.6 in Section 4.4. Applying this to the surface measure on the additive Brownian surface we get the following.

COROLLARY 4.2.7. *Let μ be the surface measure on the additive Brownian surface. Then, almost surely, (4.1.1) cannot hold whenever*

$$(4.2.3) \quad p < \frac{2kq}{2k(q-1) - 1}.$$

Equivalently,

$$(4.2.4) \quad \|\widehat{f}\|_{L^{p'}(\mu)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}$$

cannot hold whenever

$$p' > \frac{2kq}{2k+1}.$$

Setting $p = 2$ in the previous theorem (as in the Stein–Tomas estimate) gives the same estimate as in Corollary 4.2.5. The exponent coming from Theorem 3.2.6 is the conjectured sharp Fourier restriction endpoint for many manifolds and this might hint that this is the right threshold here also.

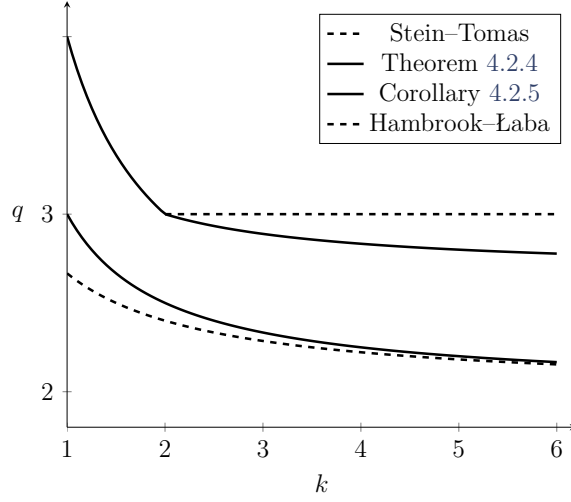


FIGURE 15. Bounds for the range of q for the Fourier extension estimate to hold and not hold for the Brownian sheet; see Theorem 4.2.4 and Corollary 4.2.5. By “Hambrook–Łaba” we refer to the observation made in [50] that no extension estimate will hold for $2 \leq q < \frac{2d}{\dim_S \mu} = \frac{2(k+1)}{k+1/2}$. These plots should be understood as only applying to integer points in the domain, but we included the full curves for aesthetic reasons. In particular, using the Fourier spectrum bounds the sharp threshold between the solid curves and appealing to previous estimates bounds the sharp threshold between the dashed curves.

4.3. Proof of Theorem 4.2.1

4.3.1. Decomposition of the Fourier transform and almost sure decay estimates.

Let $W : [0, 1]^k \rightarrow \mathbb{R}$ be the additive Brownian sheet,

$$W(t) = \sum_{i=1}^k W^i(t_i),$$

where $W^i : [0, 1] \rightarrow \mathbb{R}$ are independent 1-parameter Brownian motions. Let μ be the associated surface measure, that is, the push-forward of Lebesgue measure on $[0, 1]^k$ to W defined for $E \subset \mathbb{R}^{k+1}$ by

$$\mu(E) = \mathcal{L}^k(\{t \in [0, 1]^k : (t, W(t)) \in E\}).$$

Given $\xi \in \mathbb{R}^{k+1}$ with $\xi = (\xi_1, \dots, \xi_k, y)$, the Fourier transform decomposes as

$$\begin{aligned} \widehat{\mu}(\xi) &= \int_{[0, 1]^k} e^{-2\pi i(\xi_1 t_1 + \dots + \xi_k t_k + y W(t))} dt \\ &= \int_0^1 e^{-2\pi i(\xi_1 t_1 + y W^1(t_1))} dt_1 \dots \int_0^1 e^{-2\pi i(\xi_k t_k + y W^k(t_k))} dt_k \end{aligned}$$

$$= \widehat{\mu}_1(\xi_1, y) \cdots \widehat{\mu}_k(\xi_k, y),$$

where for each $j = 1, \dots, k$, μ_j is the surface measure on the graph of $(1, 1)$ -Brownian motion. That is, each μ_j is the push-forward of Lebesgue measure on $[0, 1]$ to the graph of the Brownian motion W^j . Fortunately, we already have good almost sure estimates for the Fourier transforms of these measures from [35] and we recall these estimates below.

In [35] the authors estimate $\widehat{\mu}_j(\xi_j, y)$ by splitting the frequency space into regions depending on the angle that (ξ_j, y) makes with the origin. We will now use those estimates, in a slightly more explicit form. For the ‘horizontal angles’ (that is, the case when $|\xi_j| \geq |y|^2$), [35, proof of Lemma 3.4], and some random $T > 0$,

$$|\widehat{\mu}_j(\xi_j, y)| \leq \left| \int_T^1 e^{-2\pi i(\xi_j t_j + y W(t_j))} dt_j \right| \lesssim |\xi_j|^{-1},$$

almost surely. Also, [35, proof of Lemma 3.6 and Lemma 3.5] gives for the same $T > 0$,

$$|\widehat{\mu}_j(\xi_j, y)| \leq \left| \int_0^T e^{-2\pi i(\xi_j t_j + y W(t_j))} dt_j \right| \lesssim |y||\xi_j|^{-1},$$

almost surely. This estimate comes from Itô calculus. Finally, the third estimate comes from the ‘vertical angles’ (that is, the case when $|\xi_j| \leq |y|^2$). In this case, [35, proof of Lemma 3.10 and Lemma 3.7] gives that almost surely

$$|\widehat{\mu}_j(\xi_j, y)| \lesssim |y|^{-1} \sqrt{\log |y|}.$$

This estimate uses ideas from Kahane’s work on Brownian images. Therefore, putting these estimates together we obtain that almost surely

$$(4.3.1) \quad \begin{aligned} |\widehat{\mu}_j(\xi_j, y)| &\lesssim \min \left\{ \max \{ |y||\xi_j|^{-1}, |\xi_j|^{-1} \}, |y|^{-1} \sqrt{\log |y|}, 1 \right\} \\ &= \begin{cases} |y||\xi_j|^{-1}, & \text{if } 1 \leq |y| \leq |y|^2 \leq |\xi_j|; \\ |\xi_j|^{-1}, & \text{if } |y| \leq 1 \leq |\xi_j|; \\ |y|^{-1} \sqrt{\log |y|}, & \text{if } \max\{|\xi_j|, 1\} \leq |y|^2; \\ 1, & \text{if } \max\{|\xi_j|, |y|\} \leq 1. \end{cases} \end{aligned}$$

The implicit constants in the above are random, but that is sufficient for what follows. The almost sure bounds in (4.3.1) are sufficient to prove the almost sure uniform bound

$$|\widehat{\mu}_j(\xi)| \lesssim |\xi|^{-1/2} \sqrt{\log |\xi|} \quad (\xi \in \mathbb{R}^2)$$

which gave the main result of [35] that the Fourier dimension of the graph of Brownian motion and the surface measure on this graph both have Fourier dimension 1. However, one can see from (4.3.1) that this ‘worst case’ decay rate happens relatively rarely and

that a much better decay

$$|\widehat{\mu}_j(\xi)| \lesssim |\xi|^{-1} \quad (\xi \in \mathbb{R}^2)$$

happens most of the time. The Fourier spectrum leverages this and quantifies how good the average Fourier decay is.

4.3.2. Proof of the lower bound.

Fix $\theta \in (0, 1]$. We may assume that $k \geq 2$ since the case $k = 1$ is dealt with by the result of [35] and the fact that the Fourier spectrum of a measure is concave (and so the affine lower bound comes for free). We need to prove that the energies $\mathcal{J}_{s,\theta}(\mu)$ are finite for appropriate values of s . To achieve this, we split the energy integral into 6 regions defined below. By symmetry, we may restrict to the case when all the coordinates of ξ are positive and assume that $0 < \xi_k < \xi_{k-1} < \dots < \xi_2 < \xi_1$. Moreover, for convenience, we assume without loss of generality that y and y^2 are distinct from all the ξ_j and from 1. The alternative occurs on a measure zero set and therefore can be omitted from the integral. Further, we assume that at least one of the coordinates of $\xi = (\xi_1, \dots, \xi_k, y)$ is strictly greater than 1, which can be done since the integral is always finite when restricted to a neighbourhood of the origin. This means we assume $\max\{\xi_1, y\} > 1$. Therefore, in the third case of (4.3.1) (i.e. when $\max\{|\xi_j|, 1\} < y^2$) it could be that $1 < y < \xi_j < y^2$, $1 < \xi_j < y < y^2$, or $\xi_j < 1 < y < y^2$.

With this, we only need to work out the different cases for each possible position of 1, y , and y^2 relative to the list of ξ_j . That is, for $j_1, j_2, j_3 \in \{0, \dots, k+1\}$ we consider the regions given by

$$0 < \xi_k < \dots < \xi_{j_3} < 1 < \xi_{j_3-1} < \dots < \xi_{j_2} < y < \xi_{j_2-1} < \dots < \xi_{j_1} < y^2 < \xi_{j_1-1} < \dots < \xi_1,$$

and

$$0 < \xi_k < \dots < \xi_{j_3} < y^2 < \xi_{j_3-1} < \dots < \xi_{j_2} < y < \xi_{j_2-1} < \dots < \xi_{j_1} < 1 < \xi_{j_1-1} < \dots < \xi_1.$$

Two cases are needed here to account for the two possible orderings $1 < y < y^2$ and $y^2 < y < 1$. We are left with a lot of cases to consider, each taking the form of an energy integral restricted to a certain region. For each integral, we need to determine conditions on s which ensure finiteness. Then the lower bound for the Fourier spectrum will be the minimum of all of these bounds, that is, we require all of the integrals to be finite simultaneously. Fortunately, we can dramatically reduce the number of cases via a linearity argument. Indeed, it is straightforward to see (and will become apparent below) that the bounds obtained for the Fourier spectrum in each case are *linear* in the variables j_1, j_2 , and j_3 . Therefore, since linear functions on an interval achieve

their extrema on the boundary, the dominant bounds will be obtained in the following boundary cases:

For $y > 1$:

Case 1. $j_1 = 0, j_2 = 0, j_3 = 1$:

$$0 < \xi_k < \cdots < \xi_1 < 1 < y < y^2.$$

Case 2. $j_1 = 0, j_2 = 1, j_3 = k + 1$:

$$1 < \xi_k < \cdots < \xi_1 < y < y^2.$$

Case 3. $j_1 = 1, j_2 = k + 1, j_3 = k + 1$:

$$1 < y < \xi_k < \cdots < \xi_1 < y^2.$$

Case 4. $j_1 = k + 1, j_2 = k + 1, j_3 = k + 1$:

$$1 < y < y^2 < \xi_k < \cdots < \xi_1.$$

For the cases where $y < 1$ recall that we assume that $\xi_1 > 1$. Therefore, we consider only:

Case 5. $j_1 = 2, j_2 = 2, j_3 = 2$:

$$0 < \xi_k < \cdots < \xi_2 < y^2 < y < 1 < \xi_1.$$

Case 6. $j_1 = k + 1, j_2 = k + 1, j_3 = k + 1$:

$$y^2 < y < 1 < \xi_k < \cdots < \xi_1.$$

We begin with the lower bound. We will split the (s, θ) -energy as follows

$$\mathcal{J}_{s,\theta}(\mu)^{\frac{1}{\theta}} \approx \int_{|\xi|>1} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s}{\theta}-k-1} d\xi \approx \sum_{\ell=1}^6 \int_{\xi \text{ in Case } \ell} |\widehat{\mu}(\xi)|^2 |\xi|^{\frac{s}{\theta}-k-1} d\xi =: \sum_{\ell=1}^6 J_{s,\theta}^{\ell}(\mu),$$

and prove that for $s < \min\{k + \frac{\theta}{2}, 2 + k\theta\}$, all of the integrals $J_{s,\theta}^1(\mu) \dots J_{s,\theta}^6(\mu)$ are finite, where $J_{s,\theta}^{\ell}(\mu)$ refers to the energy integral restricted to the region defined above in Case ℓ .

Case 1: $0 < \xi_k < \cdots < \xi_1 < 1 < y < y^2$. We will use that for $j = 1, \dots, k$, $|\widehat{\mu}_j(\xi_j, y)| \lesssim y^{-1} \sqrt{\log y}$ almost surely, and $|\xi| \approx y$. Then, almost surely,

$$\begin{aligned} J_{s,\theta}^1(\mu) &\lesssim \int_{y=1}^{\infty} y^{\frac{s}{\theta}-k-1} \int_{\xi_1=0}^1 y^{-\frac{2}{\theta}} (\log y)^{\frac{1}{\theta}} \int_{\xi_2=0}^{\xi_1} y^{-\frac{2}{\theta}} (\log y)^{\frac{1}{\theta}} \\ &\quad \cdots \int_{\xi_k=0}^{\xi_{k-1}} y^{-\frac{2}{\theta}} (\log y)^{\frac{1}{\theta}} d\xi_k \cdots d\xi_2 d\xi_1 dy \end{aligned}$$

$$\lesssim \int_{y=1}^{\infty} y^{\frac{s-2k}{\theta}-k-1} (\log y)^{\frac{k}{\theta}} dy < \infty,$$

provided $s < 2k + k\theta$.

Case 2: $1 < \xi_k < \dots < \xi_1 < y < y^2$. For $j = 1, \dots, k$, $|\hat{\mu}_j(\xi_j, y)| \lesssim y^{-1} \sqrt{\log y}$ almost surely, and $|\xi| \approx y$. Thus, almost surely,

$$\begin{aligned} J_{s,\theta}^2(\mu) &\lesssim \int_{y=1}^{\infty} y^{\frac{s}{\theta}-k-1} \int_{\xi_1=1}^y y^{-\frac{2}{\theta}} (\log y)^{\frac{1}{\theta}} \int_{\xi_2=1}^{\xi_1} y^{-\frac{2}{\theta}} (\log y)^{\frac{1}{\theta}} \\ &\quad \dots \int_{\xi_k=1}^{\xi_{k-1}} y^{-\frac{2}{\theta}} (\log y)^{\frac{1}{\theta}} d\xi_k \dots d\xi_2 d\xi_1 dy \\ &\approx \int_{y=1}^{\infty} y^{\frac{s-2k}{\theta}-1} (\log y)^{\frac{k}{\theta}} dy < \infty, \end{aligned}$$

provided $s < 2k$.

Case 3: $1 < y < \xi_k < \dots < \xi_1 < y^2$. In this case, for $j = 1, \dots, k$, $|\hat{\mu}_j(\xi_j, y)| \lesssim y^{-1} \sqrt{\log y}$ almost surely, and $|\xi| \approx \xi_1$. We may safely assume that $s > \theta$ since $J_{\theta,\theta}(\mu) < \infty$ by concavity of the Fourier spectrum for measures. Then, almost surely,

$$\begin{aligned} J_{s,\theta}^3(\mu) &\lesssim \int_{y=1}^{\infty} y^{-\frac{2k}{\theta}} (\log y)^{\frac{k}{\theta}} \int_{\xi_1=y}^{y^2} \xi_1^{\frac{s}{\theta}-k-1} \int_{\xi_2=y}^{\xi_1} \dots \int_{\xi_k=y}^{\xi_{k-1}} d\xi_k \dots d\xi_2 d\xi_1 dy \\ &\approx \int_{y=1}^{\infty} y^{-\frac{2k}{\theta}} (\log y)^{\frac{k}{\theta}} \int_{\xi_1=y}^{y^2} \xi_1^{\frac{s}{\theta}-2} d\xi_1 dy \\ &\lesssim \int_{y=1}^{\infty} y^{\frac{2s-2k}{\theta}-2} (\log y)^{\frac{k}{\theta}} dy < \infty, \end{aligned}$$

provided $s < k + \frac{\theta}{2}$.

Case 4: $1 < y < y^2 < \xi_k < \dots < \xi_1$. Now for $j = 1, \dots, k$, $|\hat{\mu}_j(\xi_j, y)| \lesssim y \xi_j^{-1}$ almost surely, and $|\xi| \approx \xi_1$. Then, almost surely,

$$\begin{aligned} J_{s,\theta}^4(\mu) &\lesssim \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1} \int_{\xi_2=1}^{\xi_1} \xi_2^{-\frac{2}{\theta}} \dots \int_{\xi_k=1}^{\xi_{k-1}} \xi_k^{-\frac{2}{\theta}} \int_{y=1}^{\xi_k^{1/2}} y^{\frac{2k}{\theta}} dy d\xi_k \dots d\xi_2 d\xi_1 \\ &\approx \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1} \int_{\xi_2=1}^{\xi_1} \xi_2^{-\frac{2}{\theta}} \dots \int_{\xi_k=1}^{\xi_{k-1}} \xi_k^{-\frac{2}{\theta} + \frac{k}{\theta} + \frac{1}{2}} d\xi_k \dots d\xi_2 d\xi_1 \\ &\approx \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1 + (1-\frac{2}{\theta})(k-1) + \frac{k}{\theta} + \frac{1}{2}} d\xi_1 \\ &= \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-k}{\theta}-\frac{3}{2}} d\xi_1 < \infty, \end{aligned}$$

provided $s < k + \frac{\theta}{2}$.

Case 5: $0 < \xi_k < \dots < \xi_2 < y^2 < y < 1 < \xi_1$. In this case we will use the bounds $|\widehat{\mu}_j(\xi_j, y)| \lesssim 1$ for $j = 2, \dots, k$ and $|\widehat{\mu}_1(\xi_1, y)| \lesssim |\xi|^{-1}$ almost surely, and $|\xi| \approx \xi_1$. This gives, almost surely,

$$J_{s,\theta}^5(\mu) \lesssim \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1} \int_{y=0}^1 \int_{\xi_2=0}^{y^2} \dots \int_{\xi_k=0}^{\xi_{k-1}} d\xi_k \dots d\xi_2 dy d\xi_1 = \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1} d\xi_1 < \infty,$$

provided $s < 2 + k\theta$.

Case 6: $0 < y^2 < y < 1 < \xi_k < \dots < \xi_1$. For $j = 1, \dots, k$, $|\widehat{\mu}_j(\xi_j, y)| \lesssim |\xi_j|^{-1}$ almost surely, and $|\xi| \approx \xi_1$. This yields, almost surely,

$$\begin{aligned} J_{s,\theta}^6(\mu) &\lesssim \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1} \int_{\xi_2=1}^{\xi_1} \xi_2^{-\frac{2}{\theta}} \dots \int_{\xi_k=1}^{\xi_{k-1}} \xi_k^{-\frac{2}{\theta}} \int_{y=0}^1 dy d\xi_k \dots d\xi_2 d\xi_1 \\ &\approx \int_{\xi_1=1}^{\infty} \xi_1^{\frac{s-2}{\theta}-k-1} d\xi_1 < \infty, \end{aligned}$$

provided $s < 2 + k\theta$.

The minimum bounds for s are obtained in Cases 3, 4, 5 and 6. Therefore, almost surely,

$$\dim_{\mathbb{F}}^{\theta} \mu \geq \min \left\{ k + \frac{\theta}{2}, 2 + k\theta \right\},$$

which completes the proof of the lower bound.

4.3.3. Proof of the upper bound.

First note that the projection of μ onto the first k coordinates is the k -dimensional Lebesgue measure $\mathcal{L}^k$. Therefore, by Proposition 2.3.2 and [36, Proposition 6.1]

$$(4.3.2) \quad \dim_{\mathbb{F}}^{\theta} \mu \leq \dim_{\mathbb{F}}^{\theta} \mathcal{L}^k + \theta = (2 + (k-1)\theta) + \theta = 2 + k\theta,$$

which proves the first upper bound.

We now turn to the second upper bound. *A priori* this should involve seeking lower bounds for the decay rate of the Fourier transform of μ , at least along certain sequences. This information is not provided in [35] and could be technically challenging to obtain. Fortunately, we found an alternative approach based on establishing that the infinitude of the usual energy ($\theta = 1$) for values of $s > k + 1/2$ comes from a region of $\mathbb{R}^d$ which has relatively small volume. We then transfer this information to the energies we use via Jensen's inequality. We believe that this approach will be useful for other problems in the future.

Let $k + \frac{1}{2} < s < 2 + k$, in which case all the energy integrals considered in the proof of the lower bound are finite apart from possibly $J_{s,1}^3(\mu)$ and $J_{s,1}^4(\mu)$. This can be seen by comparing s with the bounds coming from each case in the proof above.

Since $\dim_S \mu \leq \dim_H G(W) = k + 1/2$ we know

$$\mathcal{J}_{s,1}(\mu) = \infty.$$

Therefore,

$$\infty = \mathcal{J}_{s,1}(\mu)^{\frac{1}{\theta}} \approx (J_{s,1}^1(\mu) + \cdots + J_{s,1}^6(\mu))^{\frac{1}{\theta}} \approx (J_{s,1}^3(\mu) + J_{s,1}^4(\mu))^{\frac{1}{\theta}}.$$

We will now slightly modify the regions for Cases 3 and 4 to show that in fact, the relevant estimate comes from Case 4.

Let $0 < \varepsilon_1 < \frac{3}{k+1}$ and $k + \frac{1}{2} < s < \frac{2k-1}{2-\varepsilon_1} + 1 < 2 + k$. We change the threshold from y^2 to $y^{2-\varepsilon_1}$ and consider the following two new cases:

Case 3*: $1 < y < \xi_k < \cdots < \xi_1 < y^{2-\varepsilon_1}$.

Case 4*: $1 < y < y^{2-\varepsilon_1} < \xi_k < \cdots < \xi_1$.

We will show that for the chosen values of s , $J_{s,1}^{3*}(\mu)$ is finite. From (4.3.1) we know that for $j = 1, \dots, k$, $|\widehat{\mu}_j(\xi_j, y)| \lesssim y^{-1} \sqrt{\log y}$ almost surely, and $|\xi| \approx \xi_1$. Therefore,

$$\begin{aligned} J_{s,1}^{3*}(\mu) &\lesssim \int_{y=1}^{\infty} y^{-2k} (\log y)^k \int_{\xi_1=y}^{y^{2-\varepsilon_1}} \xi_1^{s-k-1} \int_{\xi_2=y}^{\xi_1} \cdots \int_{\xi_k=y}^{\xi_{k-1}} d\xi_k \cdots d\xi_2 d\xi_1 dy \\ &\approx \int_{y=1}^{\infty} y^{-2k} (\log y)^k \int_{\xi_1=y}^{y^{2-\varepsilon_1}} \xi_1^{s-2} d\xi_1 dy \\ &\lesssim \int_{y=1}^{\infty} y^{-2k+(2-\varepsilon_1)(s-1)} (\log y)^k dy < \infty \end{aligned}$$

since $\varepsilon_1 > 0$ and $s < \frac{2k-1}{2-\varepsilon_1} + 1$. Since $\mathcal{J}_{s,1}(\mu) = \infty$, we deduce that $J_{s,1}^{4*}(\mu) = \infty$. Note that there are intermediate cases between 3* and 4* such as

$$1 < y < \xi_k < \cdots < \xi_2 < y^{2-\varepsilon_1} < \xi_1$$

and it is necessary to establish finiteness of all of these cases too in order to justify that the infinitude of $\mathcal{J}_{s,1}(\mu)$ comes only from case 4*. However, we obtain finiteness in the intermediate cases by appealing to the linearity argument used in the proof of the lower bound (or by direct calculation).

Note that for any $\varepsilon_2 > \varepsilon_1$,

$$\begin{aligned} \int_{\xi \text{ in Case 4}^*} |\xi|^{-\left(k+\frac{1}{2-\varepsilon_2}\right)} d\xi &\approx \int_{\xi_1=1}^{\infty} \int_{\xi_2=1}^{\xi_1} \cdots \int_{\xi_k=1}^{\xi_{k-1}} \int_{y=1}^{\xi_k^{\frac{1}{2-\varepsilon_1}}} \xi_1^{-\left(k+\frac{1}{2-\varepsilon_2}\right)} dy d\xi_k \cdots d\xi_2 d\xi_1 \\ &\approx \int_{\xi_1=1}^{\infty} \xi_1^{-1+\frac{1}{2-\varepsilon_1}-\frac{1}{2-\varepsilon_2}} d\xi_1 < \infty. \end{aligned}$$

With this, let $\varepsilon_2 > \varepsilon_1$ and let $c > 0$ be such that the measure $dm(\xi) = c|\xi|^{-(k+\frac{1}{2-\varepsilon_2})} d\xi$ is a probability measure on the region given by Case 4*. Then, by Jensen's inequality,

$$\begin{aligned}
J_{s,1}^{4*}(\mu)^{\frac{1}{\theta}} &= \left(\int_{\xi \text{ in Case } 4^*} |\widehat{\mu}(\xi)|^2 |\xi|^{s-k-1} d\xi \right)^{\frac{1}{\theta}} \\
&\approx \left(\int_{\xi \text{ in Case } 4^*} |\widehat{\mu}(\xi)|^2 |\xi|^{s-1+\frac{1}{2-\varepsilon_2}} dm(\xi) \right)^{\frac{1}{\theta}} \\
&\leq \int_{\xi \text{ in Case } 4^*} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s-1+\frac{1}{2-\varepsilon_2}}{\theta}} dm(\xi) \\
&\approx \int_{\xi \text{ in Case } 4^*} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s-1+\frac{1}{2-\varepsilon_2}}{\theta}} |\xi|^{-(k+\frac{1}{2-\varepsilon_2})} d\xi \\
&= \int_{\xi \text{ in Case } 4^*} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{s-1+\theta+(1-\theta)/(2-\varepsilon_2)}{\theta}-k-1} d\xi \\
&\leq \mathcal{J}_{s-1+\theta+\frac{1-\theta}{2-\varepsilon_2}, \theta}(\mu)^{\frac{1}{\theta}}.
\end{aligned}$$

Since $J_{s,1}^{4*}(\mu) = \infty$ for $s > k + \frac{1}{2}$, then $\dim_{\mathbb{F}}^{\theta} \mu \leq k - \frac{1}{2} + \theta + \frac{1-\theta}{2-\varepsilon_2}$. Letting $\varepsilon_1, \varepsilon_2 \rightarrow 0$ yields $\dim_{\mathbb{F}}^{\theta} \mu \leq k + \frac{\theta}{2}$ as we wanted to prove. Combining this with (4.3.2) implies the desired almost sure upper bound

$$\dim_{\mathbb{F}}^{\theta} \mu \leq \min \left\{ k + \frac{\theta}{2}, 2 + k\theta \right\}$$

for all $\theta \in [0, 1]$.

4.4. A Knapp example

Recall that the Knapp example (see e.g. [83, 24]) establishes a necessary condition for restriction estimates to hold for the sphere. The idea behind it is capturing a portion of the set which will yield a bad Fourier restriction estimate, which is achieved for the sphere by choosing a small ‘cap’ which contains a large amount of measure in a relatively flat piece of the sphere. Recall that genuinely flat sets have no Fourier decay, and therefore no Fourier restriction estimates are possible for them, and so roughly flat pieces provide natural barriers to restriction holding below certain thresholds. When working with smooth manifolds, similar Knapp examples also work. For the Brownian surface, we are looking for relatively flat pieces which hold a large amount of mass, but we will do this via vertical rectangles and rely only on the Hölder exponent of Brownian motion. We now construct this example and prove Theorem 4.2.6.

4.4.1. Proof of Theorem 4.2.6.

Fix $s \in [0, 1]^k$ and $\alpha > 0$ such that f is locally α -Hölder at s . Then there exists $0 < \delta = \delta(s, \alpha) < 1$ such that for all $t \in [0, 1]^k$ with $|s - t| < \delta$,

$$|f(s) - f(t)| \lesssim |s - t|^\alpha.$$

Note that $\{(t, f(t)) : |s - t| < \delta\}$ is then contained in a box R of side-lengths $\approx \delta \times \cdots \times \delta \times \delta^\alpha$. Let $g = 1_R$ and by definition the μ mass of R is $\approx \delta^k$ (the Lebesgue measure of a δ -cube in the domain). Then

$$\|g\|_{L^p(\mu)} = \mu(R)^{\frac{1}{p}} \approx \delta^{\frac{k}{p}}.$$

Let R^* be the dual rectangle of R , that is, the rectangle centred at the origin with sides parallel to those of R and of side-lengths $\approx \delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-\alpha}$. Multiplying by $|e^{2\pi i \xi \cdot (s, f(s))}| = 1$ for $\xi \in \mathbb{R}^{k+1}$,

$$\|\widehat{g\mu}\|_{L^q(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left| \int_R e^{-2\pi i \xi \cdot (x - (s, f(s)))} d\mu(x) \right|^q d\xi \right)^{\frac{1}{q}}.$$

If $\xi \in R^*/100$ and $x \in R$, then $\Re(e^{-2\pi i \xi \cdot (x - (s, f(s)))}) \approx 1$ and $\Im(e^{-2\pi i \xi \cdot (x - (s, f(s)))}) \leq 1/10$. Since the modulus of any complex number is at least its real part,

$$\begin{aligned} \|\widehat{g\mu}\|_{L^q(\mathbb{R}^d)} &\gtrsim \left(\int_{R^*/100} \left| \int_R e^{-2\pi i \xi \cdot (x - (s, f(s)))} d\mu(x) \right|^q d\xi \right)^{\frac{1}{q}} \\ &\gtrsim \left(\int_{R^*/100} |\mu(R)|^q d\xi \right)^{\frac{1}{q}} \\ &\gtrsim \delta^k \mathcal{L}^{k+1}(R^*)^{\frac{1}{q}} \\ &\approx \delta^{\frac{k(q-1)-\alpha}{q}} \end{aligned}$$

Therefore, if we assume (4.1.1) holds, combining the estimates above yields

$$\delta^{\frac{k(q-1)-\alpha}{q}} \lesssim \|\widehat{g\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|g\|_{L^p(\mu)} \lesssim \delta^{\frac{k}{p}}$$

and therefore

$$p \geq \frac{kq}{k(q-1) - \alpha},$$

as required.

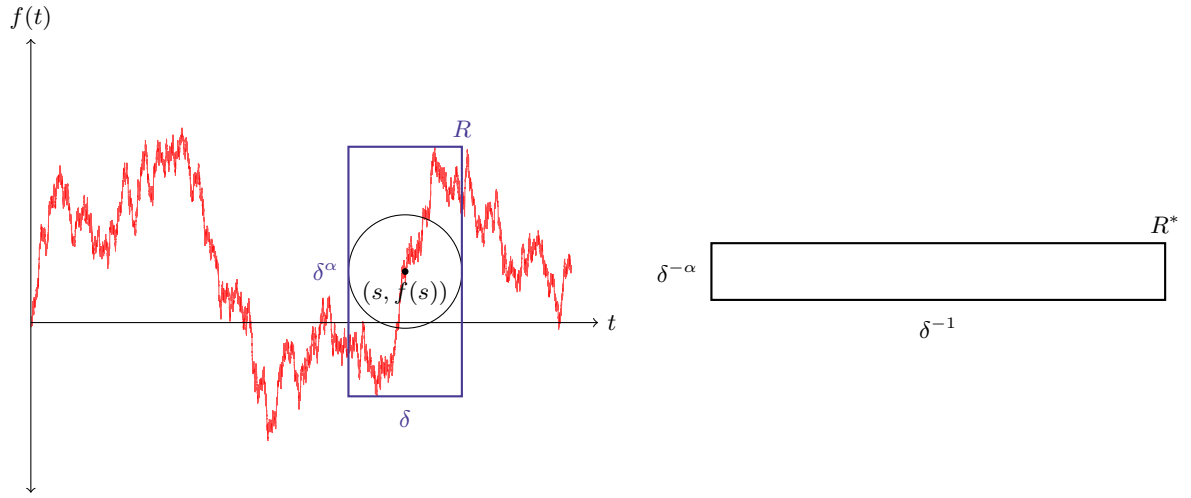


FIGURE 16. Knapp example for an α -Hölder function f , where g is the characteristic function of the rectangle R centred at $(s, f(s)) \in G(f)$ of side-lengths $\approx \delta \times \cdots \times \delta \times \delta^\alpha$. To the right is R^* the dual rectangle of R .

CHAPTER 5

Fourier restriction for multifractal measures

5.1. Introduction

The Stein–Tomas theorem, previously discussed in Chapter 3, gives the range of p for which $L^2 \rightarrow L^p$ Fourier extension estimates hold for measures satisfying a Frostman condition and having polynomial Fourier decay. Note that the notation we will use in this Chapter is different from the one used in Chapters 3 and 4, because we will reserve the variable q for the L^q -dimensions, introduced in Chapter 1. The Stein–Tomas theorem states that if $\alpha > 0$ and $\beta > 0$ are such that

$$(5.1.1) \quad \mu(B(x, r)) \lesssim r^\alpha$$

for all $x \in \mathbb{R}^d$ and $r > 0$, and

$$(5.1.2) \quad |\widehat{\mu}(\xi)|^2 \lesssim |\xi|^{-\beta}$$

for all $\xi \in \mathbb{R}^d$, then for all $f \in L^2(\mu)$, the estimate

$$(5.1.3) \quad \|\widehat{f\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$$

holds for all

$$(5.1.4) \quad p \geq 2 + 4 \frac{d - \alpha}{\beta}.$$

Or, in its dimensional version, (5.1.3) holds for all $f \in L^2(\mu)$ if

$$p > 2 + 4 \frac{d - D_\infty(\mu)}{\dim_F \mu}.$$

This result follows by interpolating between two TT^* estimates: an $L^2 \rightarrow L^2$ estimate given by the Frostman exponent, and an $L^1 \rightarrow L^\infty$ estimate given by the Fourier decay. In Chapter 3 we considered the more general Fourier decay condition given by the Fourier spectrum of μ (see Section 1.5.2), and with it obtained, for some $\theta \in [0, 1]$, an $L^{\frac{4}{4-\theta}} \rightarrow L^{\frac{4}{\theta}}$ estimate which we then interpolated with the one given by the Frostman exponent. This resulted in the range from Theorem 3.2.1,

$$(5.1.5) \quad p > 2 + 2 \inf_{\substack{\theta \in [0, 1] \\ \dim_F^\theta \mu > d\theta}} \frac{(d - D_\infty(\mu))(2 - \theta)}{\dim_F^\theta \mu - D_\infty(\mu)\theta},$$

for the estimate (5.1.3) to hold.

Furthermore, recall that in Corollary 3.5.1 we obtained a new range for p in terms only of the Fourier and Sobolev dimensions,

$$(5.1.6) \quad p > 4 + \frac{4(d - \dim_S \mu)}{\dim_F \mu},$$

which shows that for measures exhibiting multifractal behaviour (more precisely when μ satisfies $\dim_S \mu > \dim_{Fr} \mu + \frac{1}{2} \dim_F \mu$), the Mockenhaupt–Mitsis–Bak–Seeger estimate is not sharp.

As we saw in Section 1.4, the L^∞ and Sobolev (or L^2) dimensions are part of a continuum of dimensions that capture multifractality, called the L^q -dimensions. These have been widely studied in the literature of Fractal Geometry, see e.g. [56]

In this Chapter, we give a new characterisation of the L^q dimensions for $q > 1$ (see Proposition 5.3.1) that is used to obtain an $L^{p'_1} \rightarrow L^{r_1}$ estimate for the TT^* operator, and use it to interpolate with a new $L^{p'_0} \rightarrow L^{r_0}$ given by the Fourier spectrum. Interpolating between these gives a new range of p for which (5.1.3) holds in terms of the Fourier spectrum and the L^q -dimensions of the measure μ (see Theorem 5.2.1). In particular, for $q = 2$ and $\theta = 0$ we recover (5.1.6), whereas letting $q \rightarrow \infty$ we recover (5.1.5) and for $\theta = 0$ letting $q \rightarrow \infty$ recovers (5.1.4). By fixing $\theta = 0$ and letting $q \in (1, \infty]$ vary, we get a novel estimate in terms of the Fourier and L^q -dimensions. We apply this result to the Mandelbrot cascade measure in Section 5.5.

5.2. Main results

In the following two theorems we obtain a restriction estimate for fractal measures in terms of their Fourier spectrum and L^q -dimensions. Note that when $\theta = 0$ and $q = \infty$ we recover the Mockenhaupt–Mitsis–Bak–Seeger estimate (5.1.4), and letting $\theta \in [0, 1]$ and $q = \infty$ we recover Theorem 3.2.1.

THEOREM 5.2.1. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$. If for some $q > 1$ and $\theta \in [0, 1]$ such that $\dim_F^\theta \mu > d\theta$,*

$$p > 2 \frac{2(q-1)(d - D_q(\mu)) + q(\dim_F^\theta \mu - \theta d)}{(q-1)(\dim_F^\theta \mu - \theta D_q(\mu))},$$

then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, if

$$1 \leq p' < \frac{4(q-1)(d - D_q(\mu)) + 2q(\dim_F^\theta \mu - \theta d)}{4(q-1)(d - D_q(\mu)) + 2q(\dim_F^\theta \mu - \theta d) - (q-1)(\dim_F^\theta \mu - \theta D_q(\mu))},$$

then for all $f \in L^{p'}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^d)}.$$

In the previous Theorem, one can optimise over the parameters q and θ to obtain a better range for p by taking it to be

$$p > 2 \inf_{\substack{\theta \in [0,1] \\ \dim_{\mathbb{F}}^\theta \mu > d\theta \\ q > 1}} \frac{2(q-1)(d - D_q(\mu)) + q(\dim_{\mathbb{F}}^\theta \mu - \theta d)}{(q-1)(\dim_{\mathbb{F}}^\theta \mu - \theta D_q(\mu))},$$

and analogously for p' taking the supremum.

In the proof of Theorem 5.2.1 we also obtain the following endpoint estimate.

THEOREM 5.2.2. *Let $0 < \alpha < d$, $\theta \in [0, 1]$, $d\theta < \beta$, and let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$, such that for some $q > 1$*

$$\int \left(\int \frac{d\mu(y)}{|x - y|^{\frac{d+\alpha(q-1)}{q}}} \right)^q dx < \infty,$$

and

$$\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{\beta}{\theta} - d} d\xi < \infty.$$

Let

$$p_0 \geq 2 \frac{2(q-1)(d - \alpha) + q(\beta - \theta d)}{(q-1)(\beta - \theta \alpha)}.$$

Then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^{p_0}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, let

$$1 \leq p'_0 \leq \frac{4(q-1)(d - \alpha) + 2q(\beta - \theta d)}{4(q-1)(d - \alpha) + 2q(\beta - \theta d) - (q-1)(\beta - \theta \alpha)}.$$

Then for all $f \in L^{p'_0}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{p'_0}(\mathbb{R}^d)}.$$

Setting $\theta = 0$ in Theorem 5.2.1 for measures with polynomial Fourier decay, we obtain the following restriction estimate involving the Fourier and L^q -dimensions only. This result recovers Corollary 3.5.1 when $q = 2$.

COROLLARY 5.2.3. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$ with $\dim_{\mathbb{F}} \mu > 0$. If for some $q > 1$*

$$p > \frac{2q}{q-1} + \frac{4(d - D_q(\mu))}{\dim_{\mathbb{F}} \mu},$$

then for all $f \in L^2(\mu)$,

$$\|\widehat{f\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}.$$

Equivalently, if

$$1 \leq p' < 2 \frac{2(q-1)(d - D_q(\mu)) + q \dim_{\mathbb{F}} \mu}{4(q-1)(d - D_q(\mu)) + (q+1) \dim_{\mathbb{F}} \mu},$$

then for all $f \in L^{p'}(\mathbb{R}^d)$,

$$\|\widehat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^d)}.$$

A simple application of Corollary 5.2.3 gives the following bound on the L^q spectrum of measures that satisfy sharpness of the Mockenhaupt–Mitsis–Bak–Seeger theorem, e.g. those defined in [50, 51, 44, 45]. Note that since $D_{\infty}(\mu) \geq \frac{1}{2} \dim_{\mathbb{F}} \mu$, this bound is always better than the trivial

$$D_q(\mu) \leq \frac{q}{q-1} D_{\infty}(\mu)$$

Similar results will hold using Theorem 5.2.1 instead of Corollary 5.2.3.

PROPOSITION 5.2.4. *Let μ be a non-zero, finite, compactly supported, Borel measure on $\mathbb{R}^d$ satisfying (5.1.1), (5.1.2), and (5.1.3) with*

$$p = 2 + 4 \frac{d - \alpha}{\beta}.$$

Then for any $q > 1$,

$$D_{\infty}(\mu) \leq D_q(\mu) < D_{\infty}(\mu) + \frac{\dim_{\mathbb{F}} \mu}{2(q-1)}.$$

PROOF. Since μ shows sharpness of the Mockenhaupt–Mitsis–Bak–Seeger estimate, then it must be that for any $D(q) < D_q(\mu)$

$$\frac{4(d - D(q))}{\beta} + \frac{2q}{q-1} \geq 2 + 4 \frac{d - \alpha}{\beta}.$$

This gives

$$D(q) \leq \alpha + \frac{\beta}{2(q-1)},$$

which yields the desired result upon letting $\beta \rightarrow \dim_{\mathbb{F}} \mu$, $\alpha \rightarrow D_{\infty}(\mu)$, and $D(q) \rightarrow D_q(\mu)$; and recalling that for any q , $D_q(\mu) \geq D_{\infty}(\mu)$. $\square$

5.3. Obtaining the L^q dimensions from a convolution norm

To leverage the information provided by the L^q -dimensions, we use an alternative definition in terms of L^q norms, see Section 1.4. There are many alternative formulations of the L^q -dimensions in the literature (see e.g. [97]), since we could not find this version in the literature, we give a proof of a new characterisation, which may be of interest in its own right.

PROPOSITION 5.3.1. *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$ and $q > 1$. Then*

$$D_q(\mu) = \sup \left\{ s \geq 0 : \int_{\mathbb{R}^d} \left(\int \frac{d\mu(y)}{|x-y|^{\frac{d+(q-1)s}{q}}} \right)^q dx < \infty \right\}.$$

PROOF. First define

$$T(q) = \sup \left\{ s \geq 0 : \int_{\mathbb{R}^d} \left(\int \frac{d\mu(y)}{|x-y|^{\frac{d+(q-1)s}{q}}} \right)^q dx < \infty \right\}.$$

First, let $s < T(q)$ and $r \in (0, 1)$. Write Q_x for the r -cube from $\mathcal{Q}_r$ containing x . If this is not uniquely defined, then choose one arbitrarily. Then, writing $t = \frac{d+(q-1)s}{q}$ for simplicity,

$$\begin{aligned} r^{-s(q-1)} \sum_{Q \in \mathcal{Q}_r} \mu(Q)^q &= r^{-s(q-1)} \int_{\mathbb{R}^d} \mu(Q_x)^q r^{-d} dx \\ &= \int_{\mathbb{R}^d} \left(\frac{\mu(Q_x)}{r^t} \right)^q dx \\ &= \int_{\mathbb{R}^d} \left(\int_{Q_x} \frac{d\mu(y)}{r^t} \right)^q dx \\ &\lesssim \int_{\mathbb{R}^d} \left(\int_{Q_x} \frac{d\mu(y)}{|x-y|^t} \right)^q dx \\ &\leq \int_{\mathbb{R}^d} \left(\int \frac{d\mu(y)}{|x-y|^t} \right)^q dx \\ &< \infty. \end{aligned}$$

Since this holds for all $r > 0$, this proves that $D_q(\mu) \geq s$ and therefore $D_q(\mu) \geq T(q)$.

We now prove the other direction. Let $s < D_q(\mu)$, $r \in (0, 1)$ and first observe that

$$(5.3.1) \quad \int_{\mathbb{R}^d} \mu(B(x, r))^q dx \lesssim \int_{\mathbb{R}^d} \mu(Q_x)^q dx = r^d \sum_{Q \in \mathcal{Q}_r} \mu(Q)^q \lesssim r^{d+s(q-1)}.$$

Next, let $u < t = \frac{d+(q-1)s}{q}$, and observe that, for a fixed $x \in \mathbb{R}^d$,

$$\begin{aligned} \int \frac{d\mu(y)}{|x-y|^u} &= \int_{|x-y|>1} \frac{d\mu(y)}{|x-y|^u} + \sum_{k=0}^{\infty} \int_{2^{-k-1} < |x-y| \leq 2^{-k}} \frac{d\mu(y)}{|x-y|^u} \\ (5.3.2) \quad &\lesssim 1 + \sum_{k=0}^{\infty} 2^{ku} \mu(B(x, 2^{-k})). \end{aligned}$$

Finally, applying the triangle inequality for L^q norms, (5.3.1), and (5.3.2),

$$\left(\int_{\mathbb{R}^d} \left(\int \frac{d\mu(y)}{|x-y|^u} \right)^q dx \right)^{1/q} \lesssim 1 + \sum_{k=0}^{\infty} 2^{ku} \left(\int_{\mathbb{R}^d} \mu(B(x, 2^{-k}))^q dx \right)^{1/q}$$

$$\begin{aligned}
&\lesssim 1 + \sum_{k=0}^{\infty} 2^{ku} \left(2^{-k(d+s(q-1))} \right)^{1/q} \\
&= 1 + \sum_{k=0}^{\infty} 2^{k(u-t)} \\
&< \infty.
\end{aligned}$$

Letting $u \rightarrow t$, we get that $T(q) \geq s$ and therefore $T(q) \geq D_q(\mu)$, completing the proof. $\square$

5.4. Proof of Theorems 5.2.1 and 5.2.2

To prove the main theorems we will use the following lemma, known as Stein's complex interpolation theorem. We refer readers to [106, Chapter IX.1.2.5].

LEMMA 5.4.1 (Stein's complex interpolation theorem). *For $0 \leq \Re(z) \leq 1$, let T_z be a family of linear operators depending analytically on z . Let $0 < p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$ and $M_0, M_1 > 0$ be such that for $i = 0, 1$, and $\Re(z) = i$,*

$$\|T_z(f)\|_{L^{q_i}(\mathbb{R}^d)} \leq M_i \|f\|_{L^{p_i}(\mathbb{R}^d)},$$

where the constants M_i grow slower than the double exponential on $\Im(z)$. Then for all $0 \leq \lambda \leq 1$,

$$\|T_\lambda(f)\|_{L^{q_\lambda}(\mathbb{R}^d)} \leq M_0^{1-\lambda} M_1^\lambda \|f\|_{L^{p_\lambda}(\mathbb{R}^d)},$$

where $\frac{1}{p_\lambda} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$, $\frac{1}{q_\lambda} = \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}$.

Now we proceed with the proofs of Theorems 5.2.1 and 5.2.2.

Let $\theta \in (0, 1]$ and $\beta > d\theta$ such that $\dim_{\mathbb{F}}^\theta \mu > \beta$. Recall that $\kappa_s(\xi)$ denotes the Riesz kernel (see Section 1.5.1) and note that for any $a, b \in \mathbb{R}$,

$$\begin{aligned}
\|\widehat{\kappa_{a+ib} * \mu}\|_{L^{2/\theta}(\mathbb{R}^d)}^{2/\theta} &\leq \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{2(a-d)}{\theta}} |e^{i\frac{2}{\theta}b \log |\xi|}| d\xi \\
&= \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^{\frac{2}{\theta}} |\xi|^{\frac{2(a-d)}{\theta}} d\xi \\
&= \|\widehat{\kappa_a * \mu}\|_{L^{2/\theta}(\mathbb{R}^d)}^{2/\theta}.
\end{aligned}$$

Also,

$$\mathcal{J}_{s,\theta}(\mu) = \|\widehat{\kappa_{\frac{d(2-\theta)+\beta}{2}} * \mu}\|_{L^{2/\theta}(\mathbb{R}^d)}^2.$$

Then for $s_0 := \frac{d(2-\theta)+\beta}{2}$ and any $b \in \mathbb{R}$,

$$(5.4.1) \quad \|\widehat{\kappa_{s_0+ib} * \mu}\|_{L^{2/\theta}(\mathbb{R}^d)} \lesssim 1$$

Let $0 < \alpha < d$ be such that $D_q(\mu) > \alpha$. First note that for any $a, b \in \mathbb{R}$,

$$\begin{aligned} \|\kappa_{a+ib} * \mu\|_{L^q(\mathbb{R}^d)}^q &= \int_{\mathbb{R}^d} \left| \int |x-y|^{-a} e^{-ib \log |x-y|} d\mu(y) \right|^q dx \\ &\leq \int \left(\int \frac{d\mu(y)}{|x-y|^a} \right)^q dx \\ &= \|\kappa_a * \mu\|_{L^q(\mathbb{R}^d)}^q. \end{aligned}$$

Let $s_1 = \frac{d+(q-1)\alpha}{q}$, then by Proposition 5.3.1, for any $b \in \mathbb{R}$,

$$(5.4.2) \quad \|\kappa_{s_1+ib} * \mu\|_{L^q(\mathbb{R}^d)} \lesssim 1$$

For $z \in \mathbb{C}$ let $s(z) = s_0(1-z) + s_1z$ and define $T_{s(z)}f = \widehat{\kappa_{s(z)} * \mu} * f$.

Let us first consider the case $z = 0 + ib$ for $b \in \mathbb{R}$, then $s(z) = s_0 + ib(s_1 - s_0)$. By Young's inequality for convolutions and (5.4.1), if $1 + \frac{1}{r_0} = \frac{\theta}{2} + \frac{1}{p'_0}$,

$$(5.4.3) \quad \|T_{s(0+ib)}f\|_{L^{r_0}(\mathbb{R}^d)} \leq \|\widehat{\kappa_{s_0+ib(s_1-s_0)} * \mu}\|_{L^{2/\theta}(\mathbb{R}^d)} \|f\|_{L^{p'_0}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p'_0}(\mathbb{R}^d)}.$$

Now consider $z = 1 + ib$ for $b \in \mathbb{R}$, then $s(z) = s_1 + ib(s_1 - s_0)$. By the Hausdorff-Young and Hölder's inequalities, if $r_1, p_1 \geq 2$ and $\frac{1}{q} + \frac{1}{r_1} = \frac{1}{p'_1}$,

$$\begin{aligned} \|T_{s(z)}f\|_{L^{r_1}(\mathbb{R}^d)} &= \|\widehat{(\kappa_{s(z)} * \mu) \check{f}}\|_{L^{r_1}(\mathbb{R}^d)} \\ &\leq \|(\kappa_{s(z)} * \mu) \check{f}\|_{L^{r'_1}} \\ &\leq \|\kappa_{s(z)} * \mu\|_{L^q(\mathbb{R}^d)} \|\check{f}\|_{L^{p_1}(\mathbb{R}^d)} \\ (5.4.4) \quad &\leq \|\kappa_{s(z)} * \mu\|_{L^q(\mathbb{R}^d)} \|f\|_{L^{p'_1}(\mathbb{R}^d)}, \end{aligned}$$

which by (5.4.2) gives

$$(5.4.5) \quad \|T_{s(1+ib)}f\|_{L^{r_1}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p'_1}(\mathbb{R}^d)}.$$

From (5.4.3) and (5.4.5) it follows from Stein's complex interpolation theorem, Lemma 5.4.1, that for any $\lambda \in [0, 1]$,

$$\|T_{s(\lambda)}f\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^d)},$$

where

$$(5.4.6) \quad \frac{1}{r} = \frac{1-\lambda}{r_0} + \frac{\lambda}{r_1} \quad \text{and} \quad \frac{1}{p'} = \frac{1-\lambda}{p'_0} + \frac{\lambda}{p'_1}.$$

Let $\lambda = \frac{d-s_0}{s_1-s_0}$, since $\beta > d\theta$, $\lambda \in (0, 1)$, then $s(\lambda) = d$ and $T_{s(\lambda)}f = \widehat{\mu} * f$. Taking $r = p$, (5.4.6) gives

$$p = 2 \frac{q\beta - 2\alpha(q-1) - d(q\theta + 2 - 2q)}{(\beta - \theta\alpha)(q-1)},$$

which proves Theorem 5.2.2. Theorem 5.2.1 follows upon letting $\beta \rightarrow \dim_{\mathbb{F}}^{\theta} \mu$ and $\alpha \rightarrow D_q(\mu)$.

5.5. Mandelbrot Cascades

Mandelbrot cascades are an important example in fractal geometry and probability theory, and were originally introduced as a fundamental model for fluid turbulence. They are a family of random measures displaying stochastic self-similarity and have many interesting geometric properties. In particular, they are often multifractal and the Fourier dimension was recently computed (independently) in [17, 21]. These two properties make them ideal candidates for testing our theory (at least in the case $\theta = 0$). We briefly recall the definition. Let $N \geq 2$ be an integer and $W \geq 0$ be a non-negative random variable with $\mathbb{E}(W) = 1$. For each integer $n \geq 1$, let $\mathcal{D}_n$ denote the collection of closed N -adic sub-cubes of $[0, 1]^d$ of side-length N^{-n} and to each $Q \in \cup_n \mathcal{D}_n$ associate an independent copy of W , denoted by W_Q . Define a random density μ_n by

$$\mu_n(x) = \prod_{i=1}^n W_{Q_i}$$

for $x \in Q_n \in \mathcal{D}_n$ and $Q_j \in \mathcal{D}_j$ such that $Q_n \subseteq Q_j$ for each $i = 1, \dots, n$. The measures associated to these densities converge weakly to a random limit μ , which we call the Mandelbrot cascade. The Mandelbrot cascade is called non-degenerate if $\mathbb{E}(W \log W) < d \log N$ and sub-exponential if for all $t > 0$, $\mathbb{P}(W \geq t) \leq 2 \exp(-ct)$ for some c . We quote two results from the literature in the non-degenerate sub-exponential cases; see [86, 55] for the L^q -dimensions and [17, 21] for the Fourier dimension. For $p > 0$ define

$$W_p := \frac{W^p}{\mathbb{E}(W^p)},$$

and let $\lambda \in (0, \infty)$ be such that

$$\mathbb{E}(W_{\lambda} \log W_{\lambda}) = d \log N.$$

Then, almost surely conditioned on non-extinction for $q \neq 1$:

$$\tau_{\mu}(q) = \begin{cases} d - \frac{\log \mathbb{E}(W^q)}{(q-1) \log N}, & \text{if } \mathbb{E}(W_q \log W_q) \leq d \log N; \\ \frac{qd(\lambda-1)}{\lambda(q-1)} - \frac{q \log \mathbb{E}(W^{\lambda})}{\lambda(q-1) \log N}, & \text{if } \mathbb{E}(W_q \log W_q) > d \log N, \end{cases}$$

and for $q = 1$

$$\tau_{\mu}(1) = d - \frac{\mathbb{E}(W \log W)}{\log N}.$$

The Fourier dimension is given by

$$\dim_{\mathbb{F}} \mu = \min\{2, \tau_{\mu}(2)\}.$$

In particular, if we are in the subcritical regime where $\mathbb{E}(W_2 \log W_2) \leq d \log N$ and in the case $\tau_\mu(2) \leq 2$, then

$$\tau_\mu(2) = \dim_S \mu = \dim_F \mu = \dim_F^\theta \mu = d - \frac{\log \mathbb{E}(W^2)}{\log N}$$

for all $\theta \in [0, 1]$.

Applying our main result, Theorem ??, in the case $\theta = 0$ with the above in mind we get the following restriction estimate.

THEOREM 5.5.1. *Let μ be a non-degenerate, sub-exponential Mandelbrot cascade measure and $q \in (1, \infty]$ be arbitrary. Then, almost surely conditioned on non-extinction, $\|\widehat{f\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$ whenever*

$$p > \frac{2q}{q-1} + \frac{4 \log \mathbb{E}(W^q)}{(q-1)(\log N) \min\{2, \tau_\mu(2)\}}$$

if $\mathbb{E}(W_q \log W_q) \leq d \log N$ or

$$p > \frac{2q}{q-1} + \frac{4(d(q-\lambda) \log N + q \log \mathbb{E}(W^\lambda))}{\lambda(q-1)(\log N) \min\{2, \tau_\mu(2)\}}$$

if $\mathbb{E}(W_q \log W_q) > d \log N$.

In particular, if $\mathbb{E}(W_2 \log W_2) \leq d \log N$ and $\tau_\mu(2) \leq 2$, then the range becomes

$$p > \frac{2q}{q-1} + \frac{4 \log \mathbb{E}(W^q)}{(d \log N - \log \mathbb{E}(W^2))(q-1)}$$

and if $\tau_\mu(2) \geq 2$, then the range becomes

$$p > \frac{2q}{q-1} + \frac{2 \log \mathbb{E}(W^q)}{(q-1) \log N}.$$

The ranges given in the above theorem can be optimised by minimising the right-hand side as a function of the free parameter q . In the case $q = \infty$, the above reduces to the Stein–Tomas bound and it is of course interesting to consider when the bound is optimised for a different choice of q . We will see that this happens quite easily.

The restriction problem for Mandelbrot cascades in the case $d = 1$ was considered explicitly in [21, Corollary 1.6]. They proved that $\|\widehat{f\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$ whenever

$$p > \frac{4}{\dim_F \mu} = \frac{4}{\tau_\mu(2)},$$

noting that $\tau_\mu(2) \leq d < 2$ always holds in the case $d = 1$. We note that this estimate is in general not as good as the Stein–Tomas estimate and this is because the optimal value of the Frostman dimension was not used. In the subcritical case Theorem 5.5.1 recovers this range in the case $q = 2$ and in many cases will give a better bound for some $q > 2$. On the other hand, in the supercritical case (that is, whenever

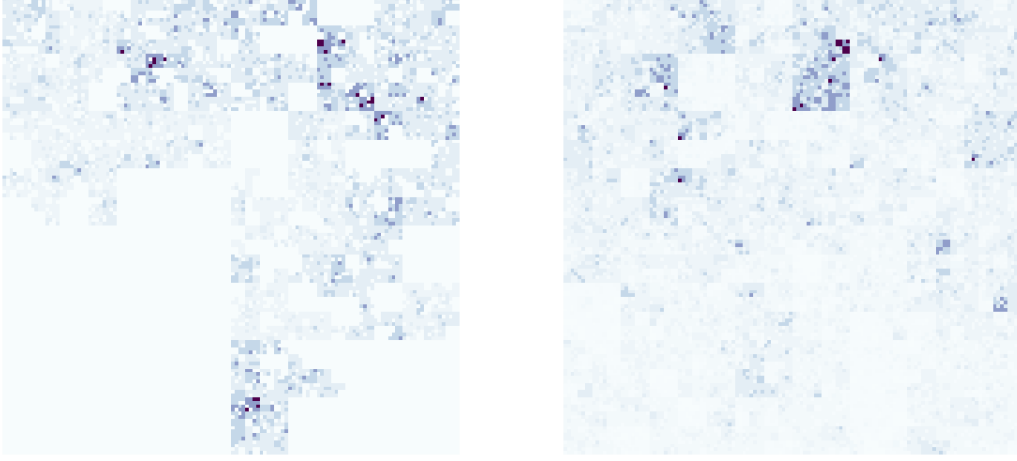


FIGURE 17. Two realisations of a Mandelbrot cascade defined with distribution (5.5.1).

$\mathbb{E}(W_2 \log W_2) > d \log N$), Theorem 5.5.1 always gives a strictly better range, even in the case $q = 2$.

We consider a simple explicit example, see Figure 17. Let $d = N = 2$ and W be defined by

$$(5.5.1) \quad W = \begin{cases} 2 & \text{with probability } 0.1 \\ 1 & \text{with probability } 0.8 \\ 0 & \text{with probability } 0.1 \end{cases}$$

Then it is easy to check that μ is non-degenerate, sub-exponential, and sub-critical and, moreover, μ is supported on a set of Hausdorff dimension $\approx 1.84 \dots < d$. Applying Theorem 5.5.1 we obtain the desired extension estimate for $p > 3.63 \dots$ and this optimal value is attained uniquely at $q \approx 4.28 \dots$. The bound at $q = 2$ is $p > 4.60 \dots$ and the bound at $q = \infty$ is $p > 3.84 \dots$, see Figure 18.

Going further, we believe it is possible to define W such that the range obtained in Theorem 5.5.1 is optimised for any given $q \in (1, \infty]$. For example, let $N = 2$, $c \in (0, 1/N)$, $u \in (c, N^d]$, and W be defined by

$$(5.5.2) \quad W = \begin{cases} u & \text{with probability } \frac{c}{u} \\ \frac{u(1-c)}{u-c} & \text{with probability } \frac{u-c}{u} \end{cases}$$

for d large enough to ensure $\tau_\mu(2) \geq 2$. Then the q that optimises the bound in Theorem 5.5.1 is unique depending on u and c . More precisely, for each $q > 1$ there exists c and large enough u and d such that the bound is optimal at q . See Figure 19 for an example where the bound can be optimised at $q = 1.34$.

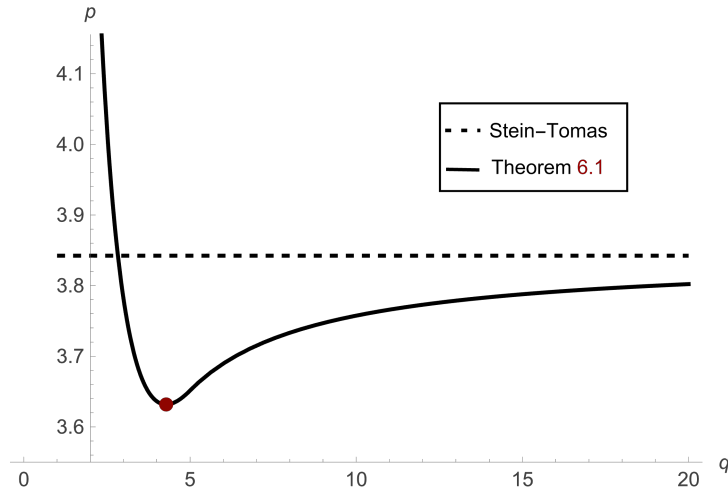


FIGURE 18. Plot of the lower bounds for p for the extension estimate to hold for the Mandelbrot cascade with distribution (5.5.1). The dashed line is the Stein–Tomas lower bound and the solid line is the one given by Theorem 5.5.1 as a function of q . The minimum value is obtained at $q \approx 4.28 \dots$

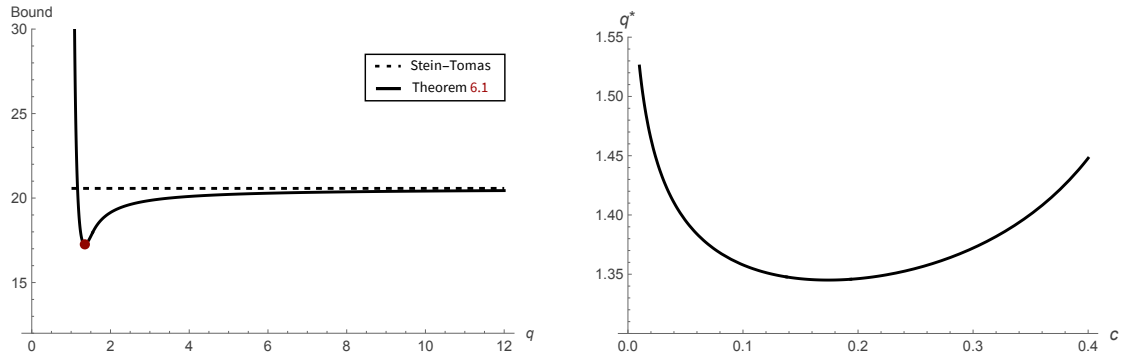


FIGURE 19. For W as in (5.5.2). Left: Bounds for $d = 11$, $N = 2$, $c = 0.22$ and $u = 1000$. The minimum value is obtained at $q = 1.34$. Right: Graph of the function which sends $c \in (0, 0.4)$ to q^* , the value of q which optimises the restriction estimate from Theorem 5.5.1, for $d = 11$, $N = 2$, and $u = 1000$.

CHAPTER 6

Multilinear Fourier Extension Estimates

6.1. Introduction

An important topic of research in harmonic analysis is concerned with extending the linear Fourier extension estimates from linear to their multilinear variants. Chapters 3, 4, and 5 have dealt with the linear Fourier extension problem, i.e. for what pairs (p, q) one has

$$(6.1.1) \quad \|\widehat{f\mu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mu)}.$$

This problem lies at the centre of a constellation of deep questions in analysis, from the study of certain Fourier summability methods to geometric measure theory and nonlinear dispersive PDEs. In Chapters 3 and 5 saw that the conditions

$$\mu(B(x, r)) \lesssim r^\alpha$$

and

$$(6.1.2) \quad |\widehat{\mu}(\xi)|^2 \lesssim |\xi|^{-\beta}$$

could be modified to an L^q -dimensional condition (as in Corollary 5.2.3), and an average Fourier decay condition, respectively.

Multilinear variants of the restriction conjecture arose in Klainerman and Machedon's work on wellposedness of certain PDEs (see [69, 70]). However, the impact these estimates had in the study of linear Fourier restriction estimates was first evidenced in [112]. The multilinear Fourier extension problem asks for which tuples of exponents $(p_1, \dots, p_k, q)$ one has

$$(6.1.3) \quad \left\| \prod_{i=1}^k \widehat{f_i \sigma_i} \right\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{i=1}^k \|f_i\|_{L^{p_i}(\sigma_i)}.$$

A straightforward application of Hölder's inequality implies a certain range of estimates such as (6.1.3) under the assumption of the linear restriction theory, and this argument gives the optimal answer to the previous question if no further assumptions on the measures σ_i (or on their supports S_i) are made.

Exploiting geometric features of the set of hypersurfaces S_i other than curvature often leads to estimates of the form (6.1.3) that are *beyond* those implied by (6.1.1)

and Hölder's inequality. We say that $k \leq d$ smooth hypersurfaces $S_1, \dots, S_k \subseteq \mathbb{R}^d$ are *transversal* if, for some $c > 0$,

$$(6.1.4) \quad |v_1 \wedge \dots \wedge v_k| \geq c$$

for all choices $v_1, \dots, v_k$ of unit normal vectors to $S_1, \dots, S_k$, respectively. In other words, if the k -dimensional volume of the parallelepiped generated by $v_1, \dots, v_k$ is bounded below by some absolute constant for any choice of normal vectors v_j , then the hypersurfaces are transversal. Under this condition, the *multilinear Fourier extension conjecture* predicts what estimates of the form (6.1.3) should hold (in the case $p_i = p_j$ for all $i, j = 1, \dots, k$):

CONJECTURE 6.1.1 ([5]). *Let $k \geq 2$ and suppose that $S_1, \dots, S_k \subseteq \mathbb{R}^d$ equipped with measures $\sigma_1, \dots, \sigma_k$, respectively, are transversal with everywhere positive principal curvatures. If $\frac{1}{q} < \frac{d-1}{2d}$, $\frac{1}{q} \leq \frac{d+k-2}{d+k} \frac{1}{p'}$ and $\frac{1}{q} \leq \frac{d-k}{d+k} \frac{1}{p'} + \frac{k-1}{k+d}$, then*

$$\left\| \prod_{i=1}^k \widehat{f_i \sigma_i} \right\|_{L^{q/k}(\mathbb{R}^d)} \lesssim \prod_{i=1}^k \|f_i\|_{L^{p_i}(\sigma_i)}$$

holds for all $f_i \in L^p(\sigma_i)$.

The curvature hypotheses in Conjecture 6.1.1 can be removed in some cases; for instance, when $k = d$ (see [5, 88] for a detailed exposition on the literature and recent progress). Conjecture 6.1.1 was settled in three cases:

- (i) $k = 2$ was done by Tao in [110] up to the endpoint.
- (ii) $k = d$ was settled by Bennett, Carbery and Tao, up to the endpoint, in [7].
- (iii) $k = d - 1$ was established by Bejenaru in [4] up to the endpoint.

A clear contrast between (6.1.1) and (6.1.3) is already seen when $k = d = 2$. If $S_1 = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ and $S_2 = \{0\} \times \mathbb{R} \subseteq \mathbb{R}^2$ are the coordinate axes of $\mathbb{R}^2$ (equipped with the induced Lebesgue measure σ_1 and σ_2 , respectively) then (6.1.1) can only hold (individually for each measure) if $q = \infty$ due to lack of curvature (or Fourier decay, see Section 1.5.1). On the other hand, it is straightforward to check that, since S_1 and S_2 are transversal in the sense of the definition above,

$$\|\widehat{f_1 \sigma_1} \widehat{f_2 \sigma_2}\|_{L^2(\mathbb{R}^2)} \lesssim \|f_1\|_{L^2(\sigma_1)} \|f_2\|_{L^2(\sigma_2)}.$$

Therefore, this is a multilinear estimate that is not obtainable directly through individual bounds on $\|\widehat{f_i \sigma_i}\|_{L^q(\mathbb{R}^2)}$, $i = 1, 2$, for any $q \geq 1$.

This raises a very natural question that is at the core of this Chapter.

QUESTION 6.1.2. *In the fractal setting, under what conditions do multilinear Fourier extension estimates hold beyond those implied by linear ones?*

As evidenced by the example above, transversality plays a fundamental role when the underlying measures are supported in smooth manifolds. In the smooth setting, transversality implies that the associated measures will have well-separated supports. This separation ensures that the wave packets (i.e. functions localised in both frequency and physical space) have limited overlaps in physical space, leading cancellation and a gain in the multilinear estimate that is unavailable in the linear theory. However, that geometric notion does not seem to have an immediate analogue in the fractal world. The aim of this Chapter is to study different manifestations of transversality in the fractal setting to obtain multilinear estimates that do not follow from linear ones.

6.1.1. First observations on multilinear estimates for fractal measures.

By the Hausdorff–Young inequality, (6.1.3) can hold for $q \leq 2$ only if the convolution of the measures at play is absolutely continuous and with an $L^{q'}$ density (this imposes $q > 2$ in the singular convolution case, see Figure 20). This is not usually a concern in the smooth setting: if σ_1 and σ_2 are two disjoint arcs of a smooth convex curve, then one can show that $\sigma_1 * \sigma_2 \ll \mathcal{L}^2$ and $\sigma_1 * \sigma_2 \in L^\infty(\mathbb{R}^2)$ (see [104]), which is enough to prove bilinear Fourier extension estimates for $(f, g) \mapsto \widehat{f\sigma_1} \widehat{g\sigma_2}$ beyond those following from linear ones. Notice that these arcs are transversal in the sense of (6.1.4), therefore bilinear estimates of that kind follow from [7], for instance. In some cases one even has, for smooth manifolds, a *characterisation* of some multilinear estimates in terms of absolute continuity of the underlying measures (see Theorem 1.1 of [6]). In general, given two smooth and transversal hypersurfaces $S_1, S_2 \subseteq \mathbb{R}^d$ equipped with measures σ_1 and σ_2 , respectively, $\sigma_1 * \sigma_2 \ll \mathcal{L}^d$. That is, transversality in the smooth setting already accounts for the convolution being absolute continuous.

The question of absolute continuity of the convolution in the fractal setting is more delicate; if $\dim_{\text{H}}(\text{spt}(\mu_1) + \text{spt}(\mu_2)) < 1$, then $\mu_1 * \mu_2$ is singular with respect to the Lebesgue measure, whereas finding conditions that guarantee $\mu_1 * \mu_2 \ll \mathcal{L}^d$ is difficult in general (see [102]). The canonical example of a fractal measure is the natural self-similar measure μ_r in the middle $(1 - 2r)$ -Cantor set for some $0 < r < \frac{1}{2}$. Studying the interaction between these measures for different r in physical space suggests that a reasonable notion of ‘transversality’ for two Cantor measures μ_{r_1} and μ_{r_2} could be the requirement that r_1 and r_2 are rationally independent, i.e. that they satisfy $\frac{\log r_1}{\log r_2} \notin \mathbb{Q}$; such heuristics, however, is not supported by the available evidence. As pointed out in [89], there is a dense G_δ set of parameters $u \in \mathbb{R}$ such that, with $T_u(x) = ux$, $\mu_{1/3} * T_u \mu_{1/4}$ is singular despite the fact that $\frac{\log 3}{\log 4} \notin \mathbb{Q}$ (it is not known whether $u = 1$ lies in the G_δ set or not). Examples like this one show that defining a suitable notion of transversality in the fractal setting will need to take into account subtleties that are

not present in classical (smooth) Fourier restriction theory. Note that in this Thesis, we refer to ‘transversality’ as the geometric condition defined in (6.1.4), which, to the best of our knowledge, is not related to transversality methods in fractal geometry (see e.g. [2]).

We first obtain a sufficient condition on the convolution of the measures so that multilinear Fourier extension estimates hold. More precisely, in Theorem 6.2.1 we show that such estimates will hold whenever $\mu_1 * \cdots * \mu_k$ is absolutely continuous with respect to $\mathcal{L}^d$ and is in some Lebesgue space. This result extends the work of [114] (based on [18]) where the author considered the stronger assumption of an L^∞ density. Second, we specialise to the dimension 1 case and find necessary conditions for multilinear Fourier extension estimates (6.1.3) with $p_i = p$ for $i = 1, \dots, k$ to hold for measures $\mu_1, \dots, \mu_k$ under certain hypotheses (see Theorems 6.2.6 and 6.2.7).

In [114, Example 5.2] the author showed that the two natural measures μ_1 and μ_2 on the IFSs

$$\begin{aligned}\varphi_1 &= \left\{ \frac{x}{4}, \frac{x}{4} + \frac{1}{4} \right\}; \\ \varphi_2 &= \left\{ \frac{x}{4}, \frac{x}{4} + \frac{1}{2} \right\},\end{aligned}$$

satisfy that $\mu_1 * \mu_2 = \mathcal{L}^1|_{[0,1]}$ (and thus this convolution is absolutely continuous and in L^∞). This naturally raises the question of whether weaker arithmetic or combinatorial conditions between the digit sets might imply that the convolution lies in some L^p space for $p \neq \infty$. Rather than pursuing digit set conditions directly, we rely on the fractal geometry literature to give examples of measures that satisfy the conditions of Theorem 6.2.1 depending on their dimensional and Fourier analytic properties. These results allow us to give many examples of measures for which multilinear Fourier extension estimates hold depending on the L^q and Fourier dimensions of the measures involved (see Sections 1.4 and 1.5.1).

In addition, building on the work of Hambrook–Łaba and Chen [50, 19], in Theorems 6.2.6 and 6.2.7 we give a necessary condition for multilinear Fourier extension estimates to hold in dimension 1. The result we obtain is more restrictive than [114, Proposition 5.3] in many cases of interest; for instance, whenever the sum of the Hausdorff dimensions of the supports of the measures involved is less than 1, and it could be complemented by a singular convolution version of Theorem 6.2.1. As shown in Example 6.2.10, multilinear Fourier extension estimates still hold for measures with singular convolution. We believe that it would be interesting to explore further the singular case and to obtain a multilinear estimate that does not follow from linear ones.

6.2. Main Results

The following theorem gives a condition on the convolution $\mu_1 * \dots * \mu_k$ of k measures so that multilinear Fourier extension estimates hold. This result, following the lines of [18, 114], recovers it when setting $p = q'$. We believe that its main value is its applicability; as we will see, it is a great source of examples.

THEOREM 6.2.1. *Let $\mu_1, \dots, \mu_k$ be finite, compactly supported, Borel measures on $\mathbb{R}^d$. If for some $p \geq 1$ and $q \geq \max(2, p')$, $\mu_1 * \dots * \mu_k \in L^{\frac{q(p-1)}{q(p-1)-p}}(\mathbb{R}^d)$, then*

$$\left\| \prod_{i=1}^k \widehat{f_i \sigma_i} \right\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{i=1}^k \|f_i\|_{L^p(\mu_i)}$$

holds for all $f_i \in L^p(\mu_i)$.

Determining the conditions that guarantee that the convolution of measures will be absolutely continuous is a fundamental question in fractal geometry, and the different notions of dimensions defined in Section 1.3 lie at its core. In [102] the authors gave an answer to this question using the L^q and Fourier dimensions of the measures involved. The following two corollaries follow from applying [102, Theorem 4.4] and [102, Theorem D] respectively, to Theorem 6.2.1.

COROLLARY 6.2.2. *Let μ, ν be two finite, compactly supported, Borel measures on $\mathbb{R}^d$. Let $p \geq 1$, $q \geq \max(2, p')$, and $p_0 := \frac{q(p-1)}{q(p-1)-p}$. If*

$$d - D_{p_0}(\mu) < \dim_F \nu$$

then

$$\|\widehat{f\mu} \widehat{g\nu}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^p(\nu)}$$

holds for all $f \in L^p(\mu)$, $g \in L^p(\nu)$.

COROLLARY 6.2.3. *For $j = 1, 2$, let $\varphi_j = (\lambda_j x + a_{ji})_{i=1}^{m_j}$ be an IFS on $\mathbb{R}$ satisfying the strong separation condition, with $\frac{\log |\lambda_2|}{\log |\lambda_1|} \notin \mathbb{Q}$ and $\frac{\log m_1}{|\log \lambda_1|} + \frac{\log m_2}{|\log \lambda_2|} > 1$. Let μ_j be a self-similar measure determined by φ_j . For $p \geq 1$, $q \geq \max(2, p')$, let $p_0 = \frac{q(p-1)}{q(p-1)-p}$. If $D_{p_0}(\mu_1) + D_{p_0}(\mu_2) > 1$, then there exists a set $E \subseteq \mathbb{R}$ of Hausdorff dimension zero such that for all $u \in \mathbb{R} \setminus E$, the estimate*

$$(6.2.1) \quad \|\widehat{f\mu_1} \widehat{gT_u\mu_2}\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mu_1)} \|g\|_{L^p(T_u\mu_2)}$$

holds for all $f \in L^p(\mu_1)$ and $g \in L^p(T_u\mu_2)$, where $T_u(x) = ux$.

Using the fact that $\dim_F(\mu_1 * \mu_2) \geq \min\{\dim_F \mu_1 + \dim_F \mu_2, d\}$ one can trivially generalise Corollary 6.2.2 to obtain k -linear estimates whenever $d - D_{p_0}(\mu_1) <$

$\sum_{i=2}^k \dim_{\mathbb{F}} \mu_i$, but we do not expect this to yield the best possible bounds. To illustrate this, suppose we are interested in proving trilinear Fourier extension estimates. Write

$$|\widehat{f_1 \mu_1} \widehat{f_2 \mu_2} \widehat{f_3 \mu_3}| = |\widehat{f_1 \mu_1} \widehat{f_2 \mu_2}|^{\frac{1}{2}} |\widehat{f_2 \mu_2} \widehat{f_3 \mu_3}|^{\frac{1}{2}} |\widehat{f_3 \mu_3} \widehat{f_1 \mu_1}|^{\frac{1}{2}}$$

and apply Hölder's inequality to bound the right-hand side by norms of bilinear operators for which either Corollary 6.2.2 or 6.2.3 can be applied. Even though we would be getting non-trivial trilinear estimates, applying Hölder's inequality destroys the simultaneous interaction between the three measures. In the smooth setting, for instance, the trilinear estimates that follow from the argument above are worse than those obtained in [7]. It would be interesting to obtain non-trivial extensions of the work in [102] to multiple convolutions, not only to provide more examples for Theorem 6.2.1, but also to shed light on new manifestations of ‘fractal transversality’ regarding multiple measures simultaneously.

The following example shows that Theorem 6.2.1 gives multilinear estimates for measures for which [114, Theorem 5.1] does not apply.

EXAMPLE 6.2.4. *Let $0 < \alpha, \beta < 1$, define the measures $\mu = x^{-\alpha} 1_{[0,1]} dx$, $\nu = x^{-\beta} 1_{[0,1]} dx$. Making the substitution $y = xt$, their convolution is*

$$\mu * \nu(x) = \int_0^x (x-y)^{-\alpha} y^{-\beta} dy = x^{1-\alpha-\beta} \int_0^1 (1-t)^{-\alpha} t^{-\beta} dt = x^{1-\alpha-\beta} B(1-\alpha, 1-\beta),$$

where $B(z_1, z_2)$ is the Beta function, which converges for $z_1, z_2 > 0$. Since $0 < \alpha, \beta < 1$, then $d(\mu * \nu) = c x^{1-\alpha-\beta} dx$. It suffices to take α and β such that $1 < \alpha + \beta < 2$. Then $\mu * \nu \in L^p(\mathbb{R})$ for any $1 \leq p < 2$ but clearly $\mu * \nu \notin L^\infty(\mathbb{R})$.

Corollary 6.2.3 is also a good source of examples, although less explicit than those following from Corollary 6.2.2.

EXAMPLE 6.2.5. *Consider the two IFSs*

$$\begin{aligned} \varphi_1 &= \left\{ \frac{1}{4}x + a_1, \frac{1}{4}x + a_2, \frac{1}{4}x + a_3 \right\}; \\ \varphi_2 &= \left\{ \frac{1}{3}x + b_1, \frac{1}{3}x + b_2 \right\}, \end{aligned}$$

where $a_i, b_i \in [0, 1]$ can be chosen arbitrarily so that both IFSs satisfy the strong separation condition. Let μ_1, μ_2 be the self-similar measures on φ_1, φ_2 with probabilities (ρ_1, ρ_2, ρ_3) and $(\gamma, 1 - \gamma)$ respectively. Note that $\frac{\log 3}{\log 4} + \frac{\log 2}{\log 3} \approx 1.42 \dots > 1$.

Let p_0 be such that

$$(6.2.2) \quad \frac{\log(\rho_1^{p_0} + \rho_2^{p_0} + \rho_3^{p_0})}{(1 - p_0) \log 4} + \frac{\log(\gamma^{p_0} + (1 - \gamma)^{p_0})}{(1 - p_0) \log 3} > 1.$$

Since the L^q -dimensions are non-increasing in q , there will be a unique value $\tilde{p}_0 = \tilde{p}_0(\rho_1, \rho_2, \rho_3, \gamma)$ such that (6.2.2) holds for all $p_0 \in [1, \tilde{p}_0]$. This will imply that (6.2.1) holds for any $p \geq 1$ and $q \geq \max(2, p')$ satisfying $1 \leq \frac{q(p-1)}{q(p-1)-p} < \tilde{p}_0$.

As an example, let $\rho_1 = 0.1$, $\rho_2 = 0.65$. For $p_0 = 2$ and any $\gamma \in [0.3, 0.7]$, there exists $E \subseteq \mathbb{R}$ with $\dim_{\mathbb{H}} E = 0$ such that for any $u \in \mathbb{R} \setminus E$,

$$\|\widehat{f\mu_1\widehat{g\nu}}\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mu_1)} \|g\|_{L^p(\nu)}$$

holds for any $q \geq 2p'$, $p \geq 1$, where ν is the self-similar measure on the IFS $T_u \circ \varphi_2 = \{\frac{u}{3}x + ub_1, \frac{u}{3}x + ub_2\}$ with probabilities $(\gamma, 1 - \gamma)$.

We next establish a necessary condition for multilinear Fourier extension estimates to hold (see Figure 20). The underlying construction should be viewed as a multilinear analogue of the fractal Knapp examples from [50, 19]. We construct measures whose mass concentrates along transversal configurations (in some sense that will be made precise later) that favour multilinear gains, which is the new technical feature that we introduce (see Section 6.4).

THEOREM 6.2.6. *Let $0 < \beta_m \leq \alpha_m < 1$ for $m = 1, \dots, k$ be such that*

$$(6.2.3) \quad \alpha_{j+1} - \frac{\beta_{j+1}}{2} \leq \alpha_j - \frac{\beta_j}{2}, \quad \text{for all } 1 \leq j \leq k-1,$$

and

$$(6.2.4) \quad \left(\alpha_k - \frac{\beta_k}{2}\right) + (k-1) \left(\alpha_1 - \frac{\beta_1}{2}\right) < 1.$$

Then there exist k Borel measures μ_m supported on compact sets E_m for $m = 1, \dots, k$ satisfying

$$(6.2.5) \quad |\widehat{\mu_m}(\xi)| \lesssim |\xi|^{-\beta_m/2},$$

and such that for every $\varepsilon > 0$, there is $0 < r_\varepsilon < \frac{1}{4}$ with

$$(6.2.6) \quad r^{\alpha_m + \varepsilon} \lesssim \mu_m(B(x, r)) \lesssim r^{\alpha_m},$$

for all $0 < r < r_\varepsilon$, $x \in E_m$, such that if

$$q < \frac{2p(1 - \sum_{m=1}^k \alpha_m) + p \sum_{m=1}^k \beta_m}{(p-1) \sum_{m=1}^k \beta_m},$$

then there exist k sequences of functions $\{f_{\ell, m}\}_{\ell \in \mathbb{N}}$, $m = 1, \dots, k$, such that

$$\frac{\|\prod_{m=1}^k \widehat{f_{\ell, m} \mu_m}\|_{L^q(\mathbb{R})}}{\prod_{m=1}^k \|f_{\ell, m}\|_{L^p(\mu_m)}} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$

The following result differs from Theorem 6.2.6 in that the measures constructed are supported on Salem sets, and requires the dimensions α_m to be of the form $\alpha_m = \log_N t$ for $t, N \in \mathbb{N}$, however, it relaxes (6.2.6) to a Frostman condition.

THEOREM 6.2.7. *For each $m = 1, \dots, k$ let $0 < \alpha_k < \dots < \alpha_1 < 1$ such that $\alpha_m = \frac{\log t_{0,m}}{\log n_0}$ for some $t_{0,m}, n_0 \in \mathbb{N}$, and $(k-1)\alpha_1 + \alpha_k < 2$. Then there exist k Borel measures μ_m supported on compact sets E_m for $m = 1, \dots, k$ satisfying*

$$|\widehat{\mu_m}(\xi)| \lesssim |\xi|^{-\beta_m/2}$$

for every $\beta_m < \alpha_m$, and

$$\mu_m(B(x, r)) \lesssim r^{\alpha_m}$$

for $x \in E_m$ and $r > 0$, such that if

$$q < \frac{p(2 - \sum_{m=1}^k \alpha_m)}{(p-1) \sum_{m=1}^k \alpha_m},$$

then there exist k sequences of functions $\{f_{\ell,m}\}_{\ell \in \mathbb{N}}$, $m = 1, \dots, k$, such that

$$\frac{\|\prod_{m=1}^k \widehat{f_{\ell,m} \mu_m}\|_{L^q(\mathbb{R}^d)}}{\prod_{m=1}^k \|f_{\ell,m}\|_{L^p(\mu_m)}} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$

We write more precise versions of Theorems 6.2.6 and 6.2.7 in Section 6.4.3, where we go into more detail about what the measures μ_m are, and we replace (6.2.6) by a weaker condition.

The following result generalises the necessary condition from [50] and Theorem 3.2.6 from linear to multilinear estimates.

PROPOSITION 6.2.8. *Let $\mu_1, \dots, \mu_k$ be finite, compactly supported, Borel measures on $\mathbb{R}^d$. If the estimate*

$$\left\| \widehat{f_1 \mu_1} \cdots \widehat{f_k \mu_k} \right\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{m=1}^k \|f_m\|_{L^p(\mu_m)}$$

holds for some $p \in [1, \infty]$, then

$$q \geq \frac{2d}{\sum_{m=1}^k \dim_{\text{BSpt}}(\mu_m)}.$$

PROOF. Let $d > s > D_2(\mu_1 * \dots * \mu_k)$, Hölder's inequality with exponents $\frac{q}{2}$ and $\frac{q}{q-2}$ gives

$$\begin{aligned} \infty = I_s(\mu_1 * \dots * \mu_k) &\approx \int_{\mathbb{R}^d} |\widehat{\mu_1} \cdots \widehat{\mu_k}(\xi)|^2 |\xi|^{s-d} d\xi \\ &\leq \|\widehat{\mu_1} \cdots \widehat{\mu_k}\|_{L^q(\mathbb{R}^d)}^2 \left(\int_{|\xi| \geq 1} |\xi|^{\frac{q(s-d)}{q-2}} d\xi \right)^{\frac{q-2}{q}}, \end{aligned}$$

where the last integral is finite if $q < \frac{2d}{s}$. Therefore, if a multilinear Fourier extension estimate holds then we must have $q \geq \frac{2d}{D_2(\mu_1 * \dots * \mu_k)}$. Since the L^q -dimensions are decreasing

$$D_2(\mu_1 * \dots * \mu_k) \leq D_0(\mu_1 * \dots * \mu_k) = \underline{\dim}_{\text{BSpt}}(\mu_1 * \dots * \mu_k) \leq \sum_{m=1}^k \overline{\dim}_{\text{BSpt}}(\mu_m).$$

Therefore, multilinear Fourier extension estimates hold only if

$$q \geq \frac{2d}{D_2(\mu_1 * \dots * \mu_k)} \geq \frac{2d}{\sum_{m=1}^k \overline{\dim}_{\text{BSpt}}(\mu_m)},$$

as we wanted to prove. $\square$

This condition (see Figure 20) can be thought of as the analogue of the condition $q > \frac{2d}{d-1}$ in Conjecture 6.1.1. Note that this ‘top lid’ condition in the smooth setting does not include the endpoint, whereas Proposition 6.2.8 does. In the smooth setting, the necessary condition follows from stronger assumptions on the measures that we do not have here (see [112, Section 2.7]).

REMARK 6.2.9. *In Proposition 5.3 of [114], Trainor gives the following necessary condition for a multilinear Fourier extension estimate to hold: let $\mu_1, \dots, \mu_k$ be compactly supported measures on $\mathbb{R}^d$ for which there exist sequences of points $\{x_{m,j}\}_{j=1}^\infty$ for each $m = 1, \dots, k$ and of radii $\{r_j\}_{j=1}^\infty$ such that $r_j \searrow 0$*

$$(6.2.7) \quad \mu_m(B(x_{m,j}, r_j)) \approx r_j^{\gamma_m}$$

for some $0 < \gamma_j \leq d$, $j = 1, \dots$. Under these conditions, if

$$\left\| \prod_{i=1}^k \widehat{f_i \mu_i} \right\|_{L^q(\mathbb{R}^d)} \leq \prod_{i=1}^k \|f_i\|_{L^p(\mu_i)}$$

holds for all $f_i \in L^p(\mu_i)$, then

$$(6.2.8) \quad q \geq \frac{dp'}{\left(\sum_{m=1}^k \gamma_m\right)}.$$

In order to compare the ranges from Theorem 6.2.6 and Remark 6.2.9 for $d = 1$, let us establish a framework under which both results fit. Let $\mu_1, \dots, \mu_k$ be k compactly supported measures in $\mathbb{R}$ such that

(H1) μ_m satisfies the *ball condition*: for $0 \leq \alpha_m \leq 1$, $1 \leq m \leq k$, given $\varepsilon > 0$, there is $r_\varepsilon < 1/4$ such that

$$r^{\alpha_m + \varepsilon} \lesssim \mu_m(B(x, r)) \lesssim r^{\alpha_m},$$

for $x \in \text{spt}(\mu_m)$ and $0 < r < r_\varepsilon$.

(H2) μ_m has *Fourier decay*: for $0 \leq \beta_m \leq 1$, $1 \leq m \leq k$,

$$|\widehat{\mu_m}(\xi)| \lesssim |\xi|^{-\beta_m/2}.$$

Theorem 6.2.6 provides an example of k measures $\mu_1, \dots, \mu_k$ satisfying (H1) and (H2). On the other hand, any measure μ_m satisfying (H1) also satisfies the hypothesis of [114, Proposition 5.3] for $\gamma_m = \alpha_m$. A direct computation shows that, under (H1) and (H2), Theorem 6.2.6 imposes more restrictive necessary conditions than [114, Proposition 5.3] in the regime $\sum_{m=1}^k \alpha_m < 1$ (see Figure 20). Note that (H1) gives $\dim_{\text{H}} \text{spt}(\mu_m) = \dim_{\text{P}} \text{spt}(\mu_m) = \overline{\dim}_{\text{BSpt}}(\mu_m) = \alpha_m$ (see the proof of Theorem 5.7 of [81]), thus $\mu_1 * \dots * \mu_k$ is singular if $\sum_{m=1}^k \alpha_m < 1$. On the other hand, in the setting of Theorem 6.2.1 (where we need this convolution to be absolutely continuous) the necessary condition obtained in [114] will be more restrictive. We stress that (6.2.4) is a technical assumption of the particular example exhibited in Theorem 6.2.6, which is why it is not included in the framework above.

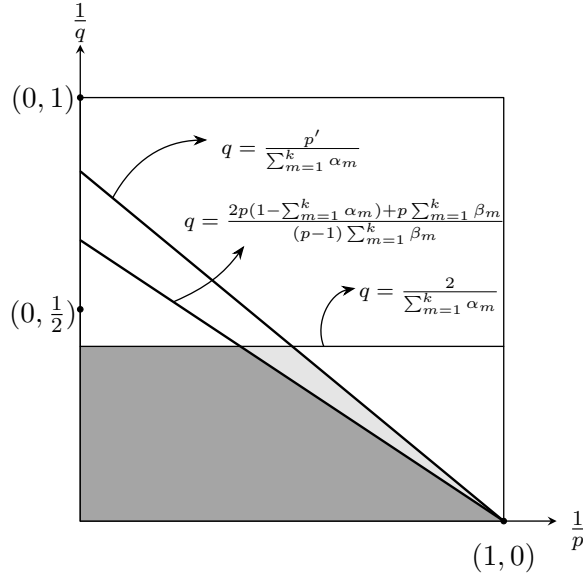


FIGURE 20. If $\sum_{m=1}^k \alpha_m < 1$, the dark grey region represents the necessary conditions of Theorem 6.2.6 intersected with $q \geq \frac{2}{\sum_{m=1}^k \alpha_m}$ from Proposition 6.2.8, since (H1) implies $\overline{\dim}_{\text{BSpt}}(\mu_m) = \alpha_m$. Notice that the latter are more restrictive than the conditions from Proposition 5.3 of [114] (represented by the union of the dark and light grey regions) in a setting with k measures $\mu_1, \dots, \mu_k$ satisfying (H1) and (H2).

A simple computation shows that, since $\beta_m \leq 2\alpha_m$ must hold (see Section 3 of [84]), the necessary condition

$$q \geq \frac{2p(1 - \sum_{m=1}^k \alpha_m) + p \sum_{m=1}^k \beta_m}{(p-1) \sum_{m=1}^k \beta_m}$$

covers more pairs (p, q) than those following from the linear theory. To illustrate this, let $k = 2$, $p = 2$ and $0 < \beta_2 = \alpha_2 \leq \alpha_1 = \beta_1 \leq 1$. Theorem 6.2.6 gives

$$q \geq \frac{4}{\alpha_1 + \alpha_2} - 2.$$

On the other hand, Theorem 3.1.1 implies

$$\left\| \widehat{f_1 \mu_1} \widehat{f_2 \mu_2} \right\|_{L^q(\mathbb{R})} \leq \left\| \widehat{f_1 \mu_1} \right\|_{L^{2q}(\mathbb{R})} \left\| \widehat{f_2 \mu_2} \right\|_{L^{2q}(\mathbb{R})} \lesssim \|f_1\|_{L^2(\mu_1)} \|f_2\|_{L^2(\mu_2)},$$

for $q \geq \frac{1}{2} \max_i \left(\frac{4}{\alpha_i} - 2 \right) = \frac{2}{\alpha_2} - 1$, hence the range

$$\frac{2}{\alpha_2} - 1 > q \geq \frac{4}{\alpha_1 + \alpha_2} - 2$$

is not covered by the linear theory and falls under the scope of Theorem 6.2.6, which suggests that there are manifestations of transversality beyond the absolute continuity of the convolution of the underlying measures (see Section 6.4.1.2).

The following example uses the linear theory to obtain a bilinear estimate for measures with singular convolution, for which we cannot apply Theorem 6.2.1.

EXAMPLE 6.2.10. *Let $0 < \beta_1 \leq \beta_2 < d$ such that $\beta_1 + \beta_2 < d$. Let E_1 be a Salem set of dimension β_1 , i.e. $\dim_F E_1 = \dim_H E_1 = \beta_1$, and E_2 be such that $\dim_H E_2 = \dim_P E_2 = \beta_2/2$. Define F_2 as the image of E_2 under Brownian motion. The set F_2 satisfies $\dim_F F_2 = \dim_H F_2 = \dim_P F_2 = \beta_2$. This is to guarantee that*

$$\dim_H(E_1 + F_2) \leq \dim_H(E_1 \times F_2) \leq \dim_H E_1 + \dim_P F_2 = \beta_1 + \beta_2 < d.$$

This way, any measure supported on $E_1 + F_2$ is singular. Let μ_1, μ_2 be measures on E_1 and F_2 respectively such that for some $\varepsilon > 0$,

$$|\widehat{\mu_1}(\xi)|^2 \lesssim |\xi|^{-(\beta_1 - \varepsilon)}; \quad |\widehat{\mu_2}(\xi)|^2 \lesssim |\xi|^{-(\beta_2 - \varepsilon)}.$$

*Since $\mu_1 * \mu_2$ is a measure supported on $E_1 + F_2$, then $\mu_1 * \mu_2$ is singular. Also, recall that the L^∞ -dimension is bounded below by half the Fourier dimension. Therefore, by Cauchy–Schwarz and the Stein–Tomas estimate, for any $q > \frac{2d}{\beta_1 - \varepsilon}$,*

$$\left\| \widehat{f \mu_1} \widehat{g \mu_2} \right\|_{L^q(\mathbb{R}^d)} \leq \left\| \widehat{f \mu_1} \right\|_{L^{2q}(\mathbb{R}^d)} \left\| \widehat{g \mu_2} \right\|_{L^{2q}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mu_1)} \|g\|_{L^2(\mu_2)}.$$

The example above shows that there are nontrivial multilinear Fourier extension estimates even if the convolution of the underlying measures is singular. This naturally raises the question of whether there is a suitable notion of ‘transversality’ between these measures whose convolution is singular (i.e. a way of exploiting their interaction) that allows one to extend the range implied by the linear theory.

6.3. Proof of Theorem 6.2.1

Let $\varphi \in \mathcal{S}_+(\mathbb{R}^d)$ be a function supported in $B(0, 1)$ with $\|\varphi\|_{L^1(\mathbb{R}^d)} = 1$, and define an approximate identity $(\varphi_\varepsilon)_{\varepsilon>0}$ by $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. For each $m = 1, \dots, k$, let $\mu_{m,\varepsilon} = \varphi_\varepsilon * \mu_m$, which converges weakly to μ_m as $\varepsilon \rightarrow 0$.

By the Hausdorff–Young inequality, since $q \geq 2$

$$\left\| \widehat{f_1 \mu_{1,\varepsilon} \cdots f_k \mu_{k,\varepsilon}} \right\|_{L^q(\mathbb{R}^d)} \leq \left\| f_1 \mu_{1,\varepsilon} * \cdots * f_k \mu_{k,\varepsilon} \right\|_{L^{q'}(\mathbb{R}^d)}.$$

Write

$$F(\xi, \eta) = f_1(\eta_1) \prod_{j=2}^k f_j(\eta_j - \eta_{j-1}); \quad M_\varepsilon = \mu_{1,\varepsilon}(\eta_1) \prod_{j=2}^k \mu_{j,\varepsilon}(\eta_j - \eta_{j-1}),$$

where $\eta = (\eta_1, \dots, \eta_{k-1}) \in \mathbb{R}^{k-1}$ and $\xi = \eta_k$. Then by the definition of convolution and Hölder's inequality for some $p \geq 1$,

$$\begin{aligned} & \left\| f_1 \mu_{1,\varepsilon} * \cdots * f_k \mu_{k,\varepsilon} \right\|_{L^{q'}(\mathbb{R}^d)} \\ &= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\xi, \eta)^{\frac{1}{p}} F(\xi, \eta) M_\varepsilon(\xi, \eta)^{\frac{1}{p'}} d\eta \right|^{q'} d\xi \right)^{\frac{1}{q'}} \\ &\leq \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\xi, \eta) d\eta \right|^{\frac{q'}{p'}} \cdot \left| \int_{\mathbb{R}^{(k-1)d}} F(\xi, \eta)^p M_\varepsilon(\xi, \eta) d\eta \right|^{\frac{q'}{p}} d\xi \right)^{\frac{1}{q'}} \\ &= \left\| \left(\int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^{(k-1)d}} F(\cdot, \eta)^p M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p}} \right\|_{L^{q'}(\mathbb{R}^d)}. \end{aligned}$$

By Hölder's inequality, with $r = \frac{pq}{q(p-1)-p}$, $\frac{1}{q'} = \frac{1}{p} + \frac{1}{r}$

$$\begin{aligned} & \left\| \left(\int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^{(k-1)d}} F(\cdot, \eta)^p M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p}} \right\|_{L^{q'}(\mathbb{R}^d)} \\ &\leq \left\| \left(\int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p'}} \right\|_{L^r(\mathbb{R}^d)} \left\| \left(\int_{\mathbb{R}^{(k-1)d}} F(\cdot, \eta)^p M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^d)} \end{aligned}$$

For the first term note that

$$\begin{aligned} \left\| \left(\int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p'}} \right\|_{L^r(\mathbb{R}^d)} &= \left\| \int_{\mathbb{R}^{(k-1)d}} M_\varepsilon(\cdot, \eta) d\eta \right\|_{L^{\frac{r}{p'}}(\mathbb{R}^d)}^{\frac{1}{p'}} \\ &= \left\| \mu_{1,\varepsilon} * \cdots * \mu_{k,\varepsilon} \right\|_{L^{\frac{r}{p'}}(\mathbb{R}^d)}^{\frac{1}{p'}} \\ &= \left\| \mu_{1,\varepsilon} * \cdots * \mu_{k,\varepsilon} \right\|_{L^{\frac{q(p-1)}{q(p-1)-p}}(\mathbb{R}^d)}^{\frac{1}{p'}}. \end{aligned}$$

Finally, the second term is

$$\left\| \left(\int_{\mathbb{R}^{(k-1)d}} F(\cdot, \eta)^p M_\varepsilon(\cdot, \eta) d\eta \right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^d)} = \prod_{i=1}^k \|f_i\|_{L^p(\mu_{i,\varepsilon})},$$

and letting $\varepsilon \rightarrow 0$ yields the desired result.

6.4. A multilinear Knapp example for random Cantor measures

Recall that the Knapp example for a smooth manifold consists of finding a function that gives a necessary condition for the linear estimate (6.1.1) by capturing flatness. The appropriate choice of such function is the characteristic function of a small cap. For simplicity, consider the sphere $(\mathbb{S}^{d-1}, \sigma_{d-1})$ in $\mathbb{R}^d$, let C_δ be a $\delta^{\frac{1}{2}} \times \dots \times \delta^{\frac{1}{2}} \times \delta$ cap of it centred at the north pole $e_d = (0, \dots, 0, 1)$ and $g := \chi_{C_\delta}$. One has

$$\frac{\|\widehat{g\sigma_{d-1}}\|_{L^q(\mathbb{R}^d)}}{\|g\|_{L^2(\sigma_{d-1})}} \gtrsim \frac{\delta^{\frac{d-1}{2} - \frac{d+1}{2q}}}{\delta^{\frac{d-1}{4}}},$$

and letting $\delta \rightarrow 0$ implies $q \geq \frac{2(d+1)}{d-1}$ as a necessary condition for an L^2 -based spherical estimate. In [50] and [19], the authors constructed Knapp-type example which proved the sharpness of the Stein–Tomas Theorem 3.1.1. The idea behind [50, 19] is to randomise the construction of a Cantor-type set such that its natural measure has Fourier decay and satisfies a ball condition. However, to understand the ‘flatness’ of the set, the construction simultaneously embeds in it an arithmetically structured Cantor set, small enough so that it does not affect these conditions. The functions that give the desired necessary condition are characteristic functions on neighbourhoods of this arithmetically structured set.

The constraints on p and q in the statement of the multilinear Conjecture 6.1.1 are more delicate: in [112], Tao, Vargas and Vega identified two necessary conditions coming from considering *squashed* and *stretched* caps. These conditions predicted that one could prove multilinear estimates beyond the range of linear ones, which were obtained shortly after by Wolff [117] (in the L^2 -based bilinear case for the cone) and Tao [110] (in the L^2 -based bilinear case for the paraboloid).

The goal of this section is to provide a multilinear analogue of Chen’s Knapp example from [19], which was based on Hambrook and Łaba’s work [50]. Generally speaking, if we were to adapt directly the ideas from [19], there would be *four* steps in the argument in which one passes from an identity to an inequality, which in turn implies a loss of information. The result presented here contains one extra technical feature compared to previous works: we require the arithmetic structures embedded in the support of each measure to be ‘linearly independent’. The latter condition will prevent us from losing information in one particular step of the argument that we see

as a manifestation of ‘transversality’, *reducing the number of lossy steps from four to three.*

6.4.1. Setup, arithmetic structure and linear independence.

Given $\phi : [2, \infty) \rightarrow (0, \infty)$ satisfying $\lim_{t \rightarrow \infty} \phi(t) = \infty$ such that $\phi(2t) \leq \phi(t) + C$ (for instance, $\phi(t) = (\log t)^\varepsilon$ for any $\varepsilon > 0$), define

$$\psi(N) = \lceil \phi(2^N)^{1/2} \rceil + 2,$$

and

$$\Psi(N) = \prod_{n=1}^N \psi(n).$$

For each $N \in \mathbb{N}$ and each $1 \leq m \leq k$, given $0 < \beta_m \leq \alpha_m < 1$, we will construct three sequences related to an arithmetic progression $W_{N,m}$.

- $\{d_{N,m}\}_{N \geq 1}$ will be the sequence of positive common differences between two consecutive elements of $W_{N,m}$.
- $\{\tau_{N,m}\}_{N \geq 1}$ will encode the number of elements of each $W_{N,m}$, i.e. $|W_{N,m}| = \tau_{N,m}$.
- $\{t_{N,m}\}_{N \geq 1}$ will be a sequence that registers the cardinality of the endpoints $A_{N,m}$ of the Cantor set at level N which will contain the arithmetic progression $W_{N,m}$.

In the construction of the Cantor-type sets E_m below, we will select a set $A_{N,m} \subseteq [\psi(N)]$ with $t_{N,m}$ elements that also contains $W_{N,m}$. Due to these constraints, the following conditions will have to be satisfied:

$$(6.4.1) \quad 1 + (\tau_{N,m} - 1)d_{N,m} \leq \psi(N),$$

and

$$(6.4.2) \quad \tau_{N,m} \leq t_{N,m} \leq \psi(N).$$

For a fixed $1 \leq m \leq k$, at each stage $N \in \mathbb{N}$ of the construction of the Cantor set E_m with associated measure μ_m , the progression

$$W_{N,m} = \{1, 1 + d_{N,m}, \dots, 1 + (\tau_{N,m} - 1)d_{N,m}\}$$

will generate the *arithmetically structured* set we need to embed inside E_m . The sequence $\{d_{N,m}\}_{N \geq 1}$ will be constructed in Lemma 6.4.3, whereas the other two are as in Chen’s work [19].

6.4.1.1. Choosing $\tau_{N,m}$ and $t_{N,m}$.

For a fixed $1 \leq m \leq k$, we choose

$$(6.4.3) \quad t_{N,m} = \psi(N)^{\alpha_m} \theta_{N,m},$$

where $\theta_{N,m}$ satisfies

$$\theta_{N,m} \in \left[\frac{1}{4}, \frac{1}{2} \right] \cup [2, 4]$$

and

$$(6.4.4) \quad \psi(N)^{\alpha_m} \log(8\Psi(N)) \approx \prod_{i=1}^{N-1} \theta_{i,m}$$

for $N \geq N_0$ large enough such that (6.4.4) holds for all $1 \leq m \leq k$. Similarly, choose

$$(6.4.5) \quad \tau_{N,m} = \psi(N)^{\alpha_m - \frac{\beta_m}{2}} \vartheta_{N,m},$$

where $\vartheta_{N,m}$ satisfies

$$\vartheta_{N,m} \in \left[\frac{1}{4}, \frac{1}{2} \right] \cup [2, 4]$$

and

$$(6.4.6) \quad \psi(N+1)^{\alpha_m} \log(8\Psi(N+1)) \approx \prod_{i=1}^N \vartheta_{i,m}$$

for $N \geq N_0$ large enough. As shown in [19], these choices are possible and they guarantee that

$$\frac{\tau_{1,m} \cdots \tau_{N,m}}{t_{1,m} \cdots t_{N,m}} \approx \Psi(N)^{-\frac{\beta_m}{2}}$$

for $N \geq N_0$, which suffices to imply the Fourier decay claimed in (6.4.13). For $N < N_0$, simply choose $\tau_{N,m} = t_{N,m} = 1$. These sequences clearly satisfy (6.4.2).

6.4.1.2. Linear Independence.

In this Section we introduce a new definition and state some properties that we will need for the construction of the multilinear Knapp-type example. We will use a construction similar to the one in [19], with k measures built containing arithmetically structured sets interacting tangentially in the following sense.

DEFINITION 6.4.1. *Let M be a positive integer. We say that the set $\{d_m\}_{m=1}^k \subseteq \mathbb{Z}$ is M -linearly independent (M -LI) if for all $\ell_i \in (-M, M) \cap \mathbb{Z}$,*

$$\sum_{m=1}^k \ell_m d_m = 0$$

if and only if $\ell_m = 0$ for all $m = 1, \dots, k$.

The following gives a simple way of constructing M -LI sets of integers.

LEMMA 6.4.2. *If for each $m = 2, \dots, k$*

$$(6.4.7) \quad d_m > M \left(\sum_{i \leq m} d_i \right)$$

then the set $\{d_m\}_{m=1}^k$ is M -LI.

PROOF. Let $\sum_{m=1}^k \ell_m d_m = 0$ with $\ell_m \in (-M, M) \cap \mathbb{Z}$. We must have $\ell_k = 0$, otherwise

$$d_k = \left| \sum_{i \leq k} \frac{\ell_i}{\ell_k} d_i \right| \leq M \left(\sum_{i \leq j} d_i \right) < d_k,$$

which contradicts (6.4.7). Continue recursively to show that $\ell_m = 0$ for all $1 \leq m \leq k$. $\square$

6.4.1.3. Choosing $d_{N,m}$.

The next lemma guarantees the maximum number of distinct sums between the different rescaled arithmetic progressions, which can be interpreted as a manifestation of transversality in this context. As mentioned in the beginning of this Section, there is one extra technical feature in the Knapp example presented in this manuscript compared to previous works. The common differences of the arithmetic progressions present in [51] and [19] play no role in their arguments, whereas we need to choose them carefully to avoid losing information here.

LEMMA 6.4.3. *For every $1 \leq m \leq k$, let α_m and β_m be given with*

$$(6.4.8) \quad 0 < \beta_m \leq \alpha_m \leq 1,$$

$$(6.4.9) \quad \alpha_{j+1} - \frac{\beta_{j+1}}{2} \leq \alpha_j - \frac{\beta_j}{2}, \quad \text{for all } 1 \leq j \leq k-1,$$

and

$$(6.4.10) \quad \left(\alpha_k - \frac{\beta_k}{2} \right) + (k-1) \left(\alpha_1 - \frac{\beta_1}{2} \right) < 1.$$

Then there exist $N_0 \in \mathbb{N}$ such that

(1) *For all $N \geq N_0$, for*

$$\tau_N := \max_{1 \leq m \leq k} \tau_{N,m},$$

and

$$M_N = \left\lceil \frac{1}{\sum_{m=1}^k \beta_m} \right\rceil (\tau_N - 1) + 1,$$

there exists an M_N -LI set $\{d_{N,m}\}_{m=1}^k$ satisfying (6.4.1).

(2) *For all $N \geq N_0$, the arithmetic progressions*

$$V_{N,m} = \{1, 1 + d_{N,m}, \dots, 1 + (M_N - 1)d_{N,m}\}$$

satisfy

$$(6.4.11) \quad |V_{N,1} + \dots + V_{N,k}| = \prod_{m=1}^k |V_{N,m}|.$$

PROOF. Choose $d_{N,m}$ satisfying $d_{N,1} = 2$ and

$$d_{N,m} = M_N \left(\sum_{k=1}^{m-1} d_k \right) + 1.$$

That is, each $d_{N,m}$ is of the form

$$d_{N,m} = (2M_N + 1)(M_N + 1)^{m-2}.$$

By Lemma 6.4.2, $\{d_{N,m}\}_{m=1}^k$ is M_N -LI. To prove (6.4.1), it suffices to show that

$$(6.4.12) \quad 1 + (\tau_{N,m} - 1)d_{N,m} = 1 + (\psi(N)^{\alpha_m - \frac{\beta_m}{2}} \vartheta_{N,m} - 1)(2M_N + 1)(M_N + 1)^{m-2} \leq \psi(N)$$

for N large enough. By (6.4.9), and our choices of $\tau_{N,m}$ and M_N , (6.4.12) holds for $N > N_0$ large enough as long as

$$\left(\alpha_m - \frac{\beta_m}{2} \right) + (m-1) \left(\alpha_1 - \frac{\beta_1}{2} \right) < 1$$

for all $m = 2, \dots, k$, which follows from (6.4.10).

To prove (6.4.11), note that there are at most $\prod_{m=1}^k |V_{N,m}|$ different sums in $V_{N,1} + \dots + V_{N,k}$. If two sums in $V_{N,1} + \dots + V_{N,k}$ coincide, i.e. if there are two different sets of coefficients $\{\ell_j\}_{j=1}^k, \{\tilde{\ell}_j\}_{j=1}^k \subseteq (-M, M) \cap \mathbb{Z}$ such that

$$\sum_{j=1}^k a_j + \sum_{j=1}^k \ell_j d_{N,j} = \sum_{j=1}^k a_j + \sum_{j=1}^k \tilde{\ell}_j d_{N,j},$$

then of course

$$\sum_{j=1}^k (\ell_j - \tilde{\ell}_j) d_j = 0.$$

Since the set $\{d_m\}_{m=1}^k$ is M -LI, we must have $\ell_j - \tilde{\ell}_j = 0$, a contradiction. We conclude that any two sums in $V_{N,1} + \dots + V_{N,k}$ are different, which immediately implies (6.4.11). $\square$

DEFINITION 6.4.4. For each $m = 1, \dots, k$, let P_m be an arithmetic progression with ratio d_m . We say that the arithmetic progressions $\{P_m\}_{m=1}^k$ are M -linearly independent (M -LI) if the corresponding set of ratios $\{d_m\}_{m=1}^k$ is M -LI.

6.4.2. Construction of Cantor sets.

Given $0 < \beta_m \leq \alpha_m < 1$, let $\{\tau_{N,m}\}_{N \in \mathbb{N}}$ and $\{t_{N,m}\}_{N \in \mathbb{N}}$ be the two sequences chosen in (6.4.3) and (6.4.5).

Recall that for $N \in \mathbb{N}$, $[N] = \{0, \dots, N-1\}$. Consider a sequence of sets $\{A_{N,m}\}_{N \in \mathbb{N}_0}$ defined in the following way

$$\begin{aligned} A_{0,m} &= \{0\}, \\ A_{N,a,m} &\subseteq [\psi(N)]/\Psi(N), \text{ for } a \in A_{N-1,m} \\ A_{N,m} &= \bigcup_{a \in A_{N-1,m}} (a + A_{N,a,m}), \\ |A_{N,a,m}| &= t_{N,m}. \end{aligned}$$

Define also

$$E_{N,m} = \bigcup_{a \in A_{N,m}} a + [0, 1/\Psi(N)], \quad E_m = \bigcap_{N=1}^{\infty} E_{N,m}.$$

E_m is a Cantor-type set with a natural probability measure μ_m defined as the weak limit of

$$\frac{d\mu_{N,m}}{dx} = \frac{1}{|E_{N,m}|} 1_{E_{N,m}}(t) dt.$$

We now continue with the construction by identifying the arithmetically structured sets $F_{N,m} \subseteq E_{N,m}$. For each N large enough, choose $\{d_{N,m}\}_{m=1}^k$ to be the $\left(\left\lceil \frac{1}{\sum_{m=1}^k \beta_m} \right\rceil (\tau_N - 1) + 1\right)$ -LI set given by Lemma 6.4.3. For each $m = 1, \dots, k$ let $W_{N,m} \subseteq [\psi(N)]$ be the following arithmetic progression of length $\tau_{N,m}$ and common difference $d_{N,m}$:

$$W_{N,m} := \{1, 1 + d_{N,m}, \dots, 1 + (\tau_{N,m} - 1)d_{N,m}\}.$$

Define the sequences of sets $P_{N,m} \subseteq A_{N,m}$, $F_{N,m} \subseteq E_{N,m}$ as

$$\begin{aligned} P_{0,m} &= \{0\}, \\ P_{N+1,m} &= \bigcup_{a \in P_{N,m}} (a + W_{N,m}/\Psi(N)), \\ F_{N,m} &= \bigcup_{a \in P_{N,m}} a + [0, 1/\Psi(N)]. \end{aligned}$$

Note that $F_{N,m} \subseteq E_{N,m}$ are smaller Cantor-type sets with endpoints in a rescaled arithmetic progression. Finally, define for each $\ell \in \mathbb{N}$

$$f_{\ell,m} = 1_{F_{\ell,m}}.$$

These functions will play the role that the characteristic function of a small cap plays in the standard Knapp example.

The construction above is similar to the one given in [19], with the additional steps of starting the construction of the sets $P_{N,m}$ with an arithmetic progression of a given common difference $d_{N,m}$, and choosing them satisfying an appropriate linear independence property. The following proposition ensures that the measures $\mu_{N,m}$ have the properties that we need.

PROPOSITION 6.4.5. *Let $\phi : [2, \infty) \rightarrow (0, \infty)$ be a non-decreasing function with $\lim_{t \rightarrow \infty} \phi(t) = \infty$ and $\phi(2t) \leq \phi(t) + C$. For each $m = 1, \dots, k$ there is a choice $A_{N,m}$ for every $N \in \mathbb{N}$ with the above properties such that for $0 < \beta_m \leq \alpha_m$,*

$$(6.4.13) \quad |\widehat{\mu}_m(\xi)| \lesssim |\xi|^{-\beta_m/2}$$

for all $\xi \in \mathbb{R} \setminus \{0\}$, and for all intervals I centred in E_m with $|I| < 1/2$,

$$(6.4.14) \quad \frac{|I|^{\alpha_m}}{\phi(1/|I|) \log(1/|I|)} \lesssim \mu(I) \lesssim \frac{|I|^{\alpha_m}}{\log(1/|I|)}.$$

What is more, if $\beta_m < \alpha_m$ the measure μ_m satisfies

$$\frac{|I|^{\alpha_m}}{\phi(1/|I|)} \lesssim \mu(I) \lesssim |I|^{\alpha_m}$$

as well as (6.4.14).

PROOF. The differences in the construction of Section 6.4.2 do not affect the results in Section 4.4 and 4.5 of [19]. Indeed, the key estimate in [19] to prove a Fourier decay inequality like our (6.4.13) depends only on the parameters $\tau_{N,m}$ and $t_{N,m}$, which we choose in the same way. As for (6.4.14), it depends only on the choice of $t_{N,m}$. $\square$

6.4.3. Proof of Theorem 6.2.6.

We start by restating Theorem 6.2.6 with more details for clarity.

Theorem 6.2.6 (in detail). *For each $m = 1, \dots, k$ let $0 \leq \beta_m \leq \alpha_m \leq 1$ be such that*

$$(6.4.15) \quad \alpha_{j+1} - \frac{\beta_{j+1}}{2} \leq \alpha_j - \frac{\beta_j}{2}, \quad \text{for all } 1 \leq j \leq k-1,$$

and

$$(6.4.16) \quad \left(\alpha_k - \frac{\beta_k}{2} \right) + (k-1) \left(\alpha_1 - \frac{\beta_1}{2} \right) < 1.$$

Let μ_m be the measures defined above, where the corresponding arithmetic progressions $\{W_{N,m}\}_{m=1}^k$ are $\left(\left\lceil \frac{1}{\sum_{m=1}^k \beta_m} \right\rceil (\tau_N - 1) + 1 \right)$ -LI for each N large enough. Let $\{f_{\ell,m}\}_{\ell \in \mathbb{N}} = \{1_{F_{\ell,m}}\}_{\ell \in \mathbb{N}}$. Assume that $p, q \in [1, \infty]$ satisfy

$$q < \frac{2p(1 - \sum_{m=1}^k \alpha_m) + p \sum_{m=1}^k \beta_m}{(p-1) \sum_{m=1}^k \beta_m}$$

Then

$$\frac{\left\| \widehat{\prod_{m=1}^k f_{\ell,m} \mu_m} \right\|_{L^q(\mathbb{R})}}{\prod_{m=1}^k \|f_{\ell,m}\|_{L^p(\mu_m)}} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$

PROOF. Throughout this proof we will assume that N is large enough as needed in Lemma 6.4.3 and Section 6.4.1.1. First note that

$$\|\widehat{f_{\ell,m} \mu_m}\|_{L^\infty(\mathbb{R})} = \mu_m(F_{\ell,m}) = \prod_{i=1}^{\ell} \frac{\tau_{i,m}}{t_{i,m}}.$$

Now we continue with some reductions. Let $r = \lceil \frac{1}{\sum_{m=1}^k \beta_m} \rceil$, then for $1 \leq q \leq 2r$,

$$(6.4.17) \quad \left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_m} \right\|_{L^q(\mathbb{R})}^q \geq \frac{\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_m} \right\|_{L^{2r}(\mathbb{R})}^{2r}}{\prod_{m=1}^k \|\widehat{f_{\ell,m} \mu_m}\|_{L^\infty(\mathbb{R})}^{2r-q}} \\ \gtrsim \prod_{i=1}^{\ell} \prod_{m=1}^k \left(\frac{t_{i,m}}{\tau_{i,m}} \right)^{2r-q} \left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_m} \right\|_{L^{2r}(\mathbb{R})}^{2r}.$$

The strategy will be to use the fact that $2r \in 2\mathbb{Z}$ and find good lower bounds for

$$\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_m} \right\|_{L^{2r}(\mathbb{R})}$$

by expanding the $L^{2r}(\mathbb{R})$ norm. Since $r > \frac{1}{\sum_{m=1}^k \beta_m}$, by Proposition 6.4.5 $|\widehat{f_{\ell,m} \mu_{N,m}}(\xi)| \lesssim g \in L^{2r}(\mathbb{R})$, and since $\widehat{f_{\ell,m} \mu_{N,m}} \rightarrow \widehat{f_{\ell,m} \mu_m}$ pointwise by the Portmanteau's theorem, we conclude that

$$\|\widehat{f_{\ell,m} \mu_{N,m}}\|_{L^{2r}(\mathbb{R})} \rightarrow \|\widehat{f_{\ell,m} \mu_m}\|_{L^{2r}(\mathbb{R})}, \quad \text{as } N \rightarrow \infty.$$

Arguing as in [19, Section 4.6], we know that for each $m = 1, \dots, k$,

$$\|f_{\ell,m}\|_{L^p(\mu_{N,m})} = \prod_{i=1}^{\ell} \left(\frac{\tau_{i,m}}{t_{i,m}} \right)^{1/p}.$$

Let $\mathcal{A}_{\ell,N,m} = F_{\ell,m} \cap A_{N,m}$. Recall that the $A_{N,m}$ are finite, therefore so are the $\mathcal{A}_{\ell,N,m}$

$$f_{\ell,m} \mu_{N,m} = \frac{1}{t_{1,m} \dots t_{N,m}} \left(\sum_{a \in \mathcal{A}_{\ell,N,m}} \delta_a \right) * \Psi(N) 1_{[0, 1/\Psi(N)]}.$$

Therefore, by developing the L^{2r} norms and by changing variables,

$$\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^{2r}(\mathbb{R})}^{2r}$$

$$\begin{aligned}
&= \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \int_{\mathbb{R}} \operatorname{sinc} \left(\frac{\xi}{\Psi(N)} \right)^{2kr} \left| \sum_{a_1 \in \mathcal{A}_{\ell,N,1}} \cdots \sum_{a_k \in \mathcal{A}_{\ell,N,k}} e^{-2\pi i \xi (a_1 + \cdots + a_k)} \right|^{2r} d\xi \\
&= \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \int_{\mathbb{R}} \operatorname{sinc} \left(\frac{\xi}{\Psi(N)} \right)^{2kr} \\
&\quad \times \left(\sum_{\substack{a_{n,1} \in \mathcal{A}_{\ell,N,1} \\ 1 \leq n \leq 2r}} \cdots \sum_{\substack{a_{n,k} \in \mathcal{A}_{\ell,N,k} \\ 1 \leq n \leq 2r}} e^{-2\pi i \xi \left(\sum_{n=1}^r \sum_{m=1}^k a_{n,m} - a_{n+r,m} \right)} \right) d\xi \\
&= \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \sum_{\substack{a_{n,1} \in \mathcal{A}_{\ell,N,1} \\ 1 \leq n \leq 2r}} \cdots \sum_{\substack{a_{n,k} \in \mathcal{A}_{\ell,N,k} \\ 1 \leq n \leq 2r}} \int_{\mathbb{R}} \operatorname{sinc} \left(\frac{\xi}{\Psi(N)} \right)^{2kr} \\
&\quad \times e^{-2\pi i \xi \left(\sum_{n=1}^r \sum_{m=1}^k a_{n,m} - a_{n+r,m} \right)} d\xi \\
&= \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \Psi(N) \sum_{\substack{a_{n,1} \in \Psi(N) \mathcal{A}_{\ell,N,1} \\ 1 \leq n \leq 2r}} \cdots \sum_{\substack{a_{n,k} \in \Psi(N) \mathcal{A}_{\ell,N,k} \\ 1 \leq n \leq 2r}} \int_{\mathbb{R}} \operatorname{sinc}(\eta)^{2kr} \\
&\quad \times e^{-2\pi i \eta \left(\sum_{n=1}^r \sum_{m=1}^k a_{n,m} - a_{n+r,m} \right)} d\eta.
\end{aligned}$$

Let $K(\eta) = \operatorname{sinc}(\eta)^{2kr}$, then

$$\begin{aligned}
\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^{2r}(\mathbb{R})}^{2r} &= \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \Psi(N) \\
&\quad \times \sum_{\substack{a_{n,1} \in \Psi(N) \mathcal{A}_{\ell,N,1} \\ 1 \leq n \leq 2r}} \cdots \sum_{\substack{a_{n,k} \in \Psi(N) \mathcal{A}_{\ell,N,k} \\ 1 \leq n \leq 2r}} \widehat{K} \left(\sum_{n=1}^r \sum_{m=1}^k a_{n,m} - a_{n+r,m} \right).
\end{aligned}$$

The following is the first step at which we potentially lose some information. We bound this sum from below by only keeping the terms that give $\widehat{K}(0)$, thus

$$\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^{2r}(\mathbb{R})}^{2r} \geq \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \Psi(N) \sum_{\substack{a_{n,1} \in \Psi(N) \mathcal{A}_{\ell,N,1} \\ \vdots \\ a_{n,k} \in \Psi(N) \mathcal{A}_{\ell,N,k} \\ 1 \leq n \leq 2r \\ \sum_{n=1}^r \sum_{m=1}^k a_{n,m} - a_{n+r,m} = 0}} \widehat{K}(0).$$

Let $C(2kr) = \widehat{K}(0) = \int_{\mathbb{R}} \operatorname{sinc}(\eta)^{2kr} d\eta$. This gives

$$\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^{2r}(\mathbb{R})}^{2r}$$

$$\begin{aligned}
&\geq C(2kr)\Psi(N) \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \\
(6.4.18) \quad &\times \left| \left\{ \begin{array}{l} (a_{1,1}, \dots, a_{2r,1}, \dots, a_{1,k}, \dots, a_{2r,k}) \in \prod_{m=1}^k (\mathcal{A}_{\ell,N,m})^{2r} \\ \text{such that} \\ \sum_{n=1}^r \sum_{m=1}^k a_{n,m} = \sum_{n=1}^r \sum_{m=1}^k a_{n+r,m} \end{array} \right\} \right|
\end{aligned}$$

by homogeneity of the relation $\sum_{n=1}^r \sum_{m=1}^k a_{n,m} = \sum_{n=1}^r \sum_{m=1}^k a_{n+r,m}$. For each $\ell, N \in \mathbb{N}$ define

$$M_{\ell,N,r} = \left| \left\{ \begin{array}{l} (a_{1,1}, \dots, a_{2r,1}, \dots, a_{1,k}, \dots, a_{2r,k}) \in \prod_{m=1}^k (\mathcal{A}_{\ell,N,m})^{2r} \\ \text{such that} \\ \sum_{n=1}^r \sum_{m=1}^k a_{n,m} = \sum_{n=1}^r \sum_{m=1}^k a_{n+r,m} \end{array} \right\} \right|.$$

We want to bound $M_{\ell,N,r}$ from below. For this, define for each $\ell, N \in \mathbb{N}$,

$$Z_{\ell,N,r} = \left\{ \sum_{n=1}^r \sum_{m=1}^k a_{n,m} : a_{n,m} \in \mathcal{A}_{\ell,N,m}, \quad \forall 1 \leq n \leq r \right\}.$$

Observe that $y_m \in \mathcal{A}_{\ell,N,m}$ has a digit representation

$$y_m = \sum_{i=1}^{\ell} \frac{y_m^{(i)}}{\Psi(i)} + \frac{y_m^{(\ell+1)}}{\Psi(N)},$$

where $y_m^{(i)} \in W_{i,m}$ for $i = 1, \dots, \ell$ and $W_{i,m} = \{1, 1 + d_{i,m}, \dots, 1 + (\tau_{N,m} - 1)d_{N,m}\}$ with $\{d_{N,m}\}_{m=1}^k$ the $\left(\left\lceil \frac{1}{\sum_{m=1}^k \beta_m} \right\rceil (\tau_N - 1) + 1 \right)$ -LI set; and $y_m^{(\ell+1)} \in [\psi(\ell+1) \cdots \psi(N)]$. Then, each $z \in Z_{\ell,N,r}$ can be written as

$$z = \sum_{i=1}^{\ell} \frac{z^{(i)}}{\Psi(i)} + \frac{z^{(\ell+1)}}{\Psi(N)}$$

with $z^{(i)} \in W'_{i,1} + \cdots + W'_{i,k}$ for $i = 1, \dots, \ell$ and $W'_{j,m} = \{r, r + d_{j,m}, \dots, r + r(\tau_{i,m} - 1)d_{i,m}\}$; and $z^{(\ell+1)} \in [kr\psi(\ell+1) \cdots \psi(N)]$.

Therefore, by Lemma 6.4.3 with $V_{N,m} = W'_{N,m}$,

$$\begin{aligned}
|Z_{\ell,N,r}| &\leq |W'_{1,1} + \cdots + W'_{1,k}| \cdots |W'_{\ell,1} + \cdots + W'_{\ell,k}| |[r\psi(\ell+1) \cdots \psi(N)]| \\
&= \prod_{i=1}^{\ell} \prod_{m=1}^k (r(\tau_{i,m} - 1) + 1) (r\psi(\ell+1) \cdots \psi(N)).
\end{aligned}$$

The inequality above comes exclusively from saying the set of possible $z^{(\ell+1)}$ is in $[kr\psi(\ell+1) \cdots \psi(N)]$, which is the second potentially lossy step. We do not lose any information with the terms $|W'_{i,1} + \cdots + W'_{i,k}|$, $1 \leq i \leq \ell$, because we compute them

explicitly. For $z \in Z_{\ell,N,r}$, define

$$g(z) = \left| \left\{ (a_{1,1}, \dots, a_{1,r}, \dots, a_{k,1}, \dots, a_{k,r}) \in \prod_{m=1}^k (\mathcal{A}_{\ell,N,m})^r : \sum_{n=1}^r \sum_{m=1}^k a_{n,m} = z \right\} \right|.$$

Then $\|g\|_{\ell^1(Z_{\ell,N,r})} = \prod_{m=1}^k |\mathcal{A}_{\ell,N,m}|^r = \prod_{m=1}^k (\tau_{1,m} \cdots \tau_{\ell,m} t_{\ell+1,m} \cdots t_{N,m})^r$ and

$$\begin{aligned} & \|g\|_{\ell^2}^2 \\ &= \sum_z \left| \left\{ (a_{1,1}, \dots, a_{1,r}, \dots, a_{k,1}, \dots, a_{k,r}) \in \prod_{m=1}^k (\mathcal{A}_{\ell,N,m})^r : \sum_{n=1}^r \sum_{m=1}^k a_{n,m} = z \right\} \right|^2 \\ &= \sum_z \left| \left\{ (a_{1,1}, \dots, a_{1,2r}, \dots, a_{k,1}, \dots, a_{k,2r}) \in \prod_{m=1}^k (\mathcal{A}_{\ell,N,m})^{2r} : \right. \right. \\ & \quad \left. \left. \sum_{n=1}^r \sum_{m=1}^k a_{n,m} = \sum_{n=1}^r \sum_{m=1}^k a_{n+r,m} = z \right\} \right| \\ &= M_{\ell,N,r}. \end{aligned}$$

This way, by Cauchy–Schwarz (which is the third potentially lossy step),

$$M_{\ell,N,r} \geq \frac{\|g\|_{\ell^1(Z_{\ell,N,r})}^2}{|Z_{\ell,N,r}|} \geq \frac{\prod_{m=1}^k (\tau_{1,m} \cdots \tau_{\ell,m} t_{\ell+1,m} \cdots t_{N,m})^{2r}}{\prod_{i=1}^{\ell} \prod_{m=1}^k (r(\tau_{i,m} - 1) + 1)(r\psi(\ell+1) \cdots \psi(N))}.$$

Expanding $\prod_{i=1}^{\ell} \prod_{m=1}^k (r(\tau_{i,m} - 1) + 1)$,

$$\prod_{i=1}^{\ell} \prod_{m=1}^k (r(\tau_{i,m} - 1) + 1) = r^{k\ell} \prod_{i=1}^{\ell} \prod_{m=1}^k \tau_{i,m} + L_{<k},$$

where $L_{<k}$ is a lower order term given by a linear combination of products of less than k factors of the form $r\tau_{i,m}$ and r . Thus,

$$M_{\ell,N,r} \geq \frac{\prod_{m=1}^k (\tau_{1,m} \cdots \tau_{\ell,m} t_{\ell+1,m} \cdots t_{N,m})^{2r}}{r^{k\ell} \prod_{i=1}^{\ell} \prod_{m=1}^k \tau_{i,m} + L_{<k}} \frac{\Psi(\ell)}{r\Psi(N)}.$$

Therefore, by (6.4.18),

$$\begin{aligned} & \left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^{2r}(\mathbb{R})}^{2r} \\ & \geq C(2kr)\Psi(N) \left(\prod_{m=1}^k \frac{1}{t_{1,m} \cdots t_{N,m}} \right)^{2r} \frac{\prod_{m=1}^k (\tau_{1,m} \cdots \tau_{\ell,m} t_{\ell+1,m} \cdots t_{N,m})^{2r}}{r^{k\ell} \prod_{i=1}^{\ell} \prod_{m=1}^k \tau_{i,m} + L_{<k}} \frac{\Psi(\ell)}{r\Psi(N)} \\ & = C(2kr)\Psi(\ell) \frac{\prod_{i=1}^{\ell} \prod_{m=1}^k (\tau_{i,m}^{2r} t_{i,m}^{-2r})}{r^{k\ell+1} \prod_{i=1}^{\ell} \prod_{m=1}^k \tau_{i,m} + rL_{<k}}. \end{aligned}$$

From (6.4.17), for $1 \leq q \leq 2r$,

$$\begin{aligned} & \left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^q(\mathbb{R})}^q \\ & \gtrsim \prod_{i=1}^{\ell} \prod_{m=1}^k \left(\frac{t_{i,m}}{\tau_{i,m}} \right)^{2r-q} \left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_m} \right\|_{L^{2r}(\mathbb{R})}^{2r} \\ & \geq C(2kr) \Psi(\ell) \prod_{i=1}^{\ell} \prod_{m=1}^k \left(\frac{t_{i,m}}{\tau_{i,m}} \right)^{2r-q} \frac{\prod_{i=1}^{\ell} \prod_{m=1}^k (\tau_{i,m}^{2r} t_{i,m}^{-2r})}{r^{k\ell+1} \prod_{i=1}^{\ell} \prod_{m=1}^k \tau_{i,m} + r L_{<k}}. \end{aligned}$$

We conclude that

$$\begin{aligned} & \frac{\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^q(\mathbb{R})}^q}{\prod_{m=1}^k \|f_{\ell,m}\|_{L^p(\mu_{N,m})}^q} \\ & \geq C(2kr) \Psi(\ell) \prod_{i=1}^{\ell} \prod_{m=1}^k \left(\frac{t_{i,m}}{\tau_{i,m}} \right)^{\frac{q}{p}-q} \left(r^{k\ell+1} \prod_{i=1}^{\ell} \prod_{m=1}^k \tau_{i,m} + r L_{<k} \right)^{-1} \\ & = \frac{C(2kr) \Psi(\ell)}{r^{k\ell+1} \prod_{i=1}^{\ell} \prod_{m=1}^k (\tau_{i,m}^{1+\frac{q}{p}-q} t_{i,m}^{q-\frac{q}{p}}) + r L_{<k} \prod_{i=1}^{\ell} \prod_{m=1}^k (\tau_{i,m}^{\frac{q}{p}-q} t_{i,m}^{q-\frac{q}{p}})}. \end{aligned}$$

Using the choices made in Section 6.4.1.1 yields

$$(6.4.19) \quad \frac{\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_{N,m}} \right\|_{L^q(\mathbb{R})}^q}{\prod_{m=1}^k \|f_{\ell,m}\|_{L^p(\mu_{N,m})}^q} \gtrsim \Gamma(\ell)^{-1},$$

where

$$\begin{aligned} \Gamma(\ell) := & r^{k\ell+1} \Psi(\ell)^{-1+\sum_{m=1}^{\ell} \alpha_m - \frac{1}{2}} \left(1 + \frac{q}{p} - q \right)^{\sum_{m=1}^k \beta_m} \psi(\ell+1)^{\sum_{m=1}^k \alpha_m} \log^k(8\Psi(\ell+1)) \\ & + r \Psi(\ell)^{-1} L_{<k} \prod_{i=1}^{\ell} \prod_{m=1}^k (\tau_{i,m}^{\frac{q}{p}-q} t_{i,m}^{q-\frac{q}{p}}). \end{aligned}$$

Because of our choices of ψ and Ψ , the right-hand side of (6.4.19) diverges as $\ell \rightarrow \infty$ if

$$1 - \sum_{m=1}^k \alpha_m + \frac{1}{2} \sum_{m=1}^{\infty} \beta_m \geq \frac{q}{2} \left(1 - \frac{1}{p} \right) \sum_{m=1}^k \beta_m,$$

which results in

$$q < \frac{2p(1 - \sum_{m=1}^k \alpha_m) + p \sum_{m=1}^k \beta_m}{(p-1) \sum_{m=1}^k \beta_m},$$

as we wanted to show. $\square$

6.4.4. Second construction of Cantor sets.

As in the previous Sections, the construction below will be a particular example of the one in [50], with the additional step of considering M -LI APs, which can be thought of as a manifestation of transversality between the measures.

Let N_0 be an integer. For each $m = 1, \dots, k$ let $1 < t_{0,m} < N_0$ be an integer such that for $\alpha_m = \log t_{0,m} / \log N_0$, $(k-1)\alpha_1 + \alpha_k < 2$. Let $N = N_0^{2n_0}$, $t_m = t_{0,m}^{2n_0}$ for some n_0 large enough to be chosen later.

For each $m = 1, \dots, k$, consider a sequence of sets $\{A_{j,m}\}_{j \in \mathbb{N}_0}$ such that

$$\begin{aligned} A_{0,m} &= \{0\}, \\ A_{j+1,a,m} &\subseteq N^{-(j+1)}[N], \text{ for } a \in A_{j,m}, \\ A_{j+1,m} &= \bigcup_{a \in A_{j,m}} (a + A_{j+1,a,m}), \\ |A_{j+1,a,m}| &= t_m. \end{aligned}$$

Define also

$$E_{j,m} = \bigcup_{a \in A_{j,m}} a + [0, N^{-j}], \quad E_m = \bigcap_{j=1}^{\infty} E_{j,m}.$$

E_m is a Cantor-type set with a natural probability measure μ_m defined as the weak limit of

$$\frac{d\mu_{j,m}}{dx} = \sum_{a \in A_{j,m}} N^{-j\alpha_m} N^j 1_{[a, a+N^{-j}]}(t) dt.$$

By a standard argument (see Lemma 6.1 of [71]), the sets E_m defined above satisfy $\dim_{\mathbb{H}} E_m = \alpha_m$ and each probability measure μ_m satisfies

$$\mu_m(B(x, r)) \lesssim_{n_0} r^{\alpha_m}$$

for any $x \in E_m$ and $r > 0$.

We now continue with the construction by identifying the arithmetically structured sets $F_{j,m} \subseteq E_{j,m}$. Since these will be given by M -LI APs, we need the following lemma that plays the same role as Lemma 6.4.3.

LEMMA 6.4.6. *Let $0 < \alpha_k < \dots < \alpha_1 < 1$, with $\alpha_m = \frac{\log t_{0,m}}{\log N_0}$ for some integers $t_{0,m}$ and N_0 , $t_{0,m} \leq N_0$ for each $1 \leq m \leq k$, and $(k-1)\alpha_1 + \alpha_k < 2$. Then*

(1) *There exist $n_0 \in \mathbb{N}$ and a set $\{d_m\}_{m=1}^k$ satisfying*

$$(6.4.20) \quad d_m \leq N_0^{2n_0(1-\frac{\alpha_m}{2})}$$

$m = 1, \dots, k$, that is $\left(\left\lceil \frac{1}{\sum_{m=1}^k \alpha_m} \right\rceil (N_0^{n_0 \alpha_1} - 1) + 1\right)$ -LI.

(2) *For any $a_i \in \mathbb{Z}$, the arithmetic progressions*

$$V_m = \left\{ a_m, a_m + d_m, \dots, a_m + \left\lceil \frac{1}{\sum_{m=1}^k \alpha_m} \right\rceil (N_0^{n_0 \alpha_1} - 1) d_m \right\}$$

$m = 1, \dots, k$, satisfy

$$(6.4.21) \quad |V_1 + \dots + V_k| = \prod_{m=1}^k |V_m|.$$

Let $\{d_m\}_{m=1}^k$ be the $\left(\left\lceil \frac{1}{\sum_{m=1}^k \alpha_m} \right\rceil (N_0^{n_0 \alpha_1} - 1) + 1\right)$ -LI set given by Lemma 6.4.6. For each $m = 1, \dots, k$ let $P_m \subseteq [N]$ be an arithmetic progression of length $t_m^{1/2} = N^{\alpha_m/2}$ and ratio d_m , that is

$$P_m = \{x_m, x_m + d_m, x_m + 2d_m, \dots, x_m + (N^{\frac{\alpha_m}{2}} - 1)d_m\}$$

for some $x_m \in [N]$. Define the sequences of sets $P_{j,m} \subseteq A_{j,m}$, $F_{j,m} \subseteq E_{j,m}$ as

$$\begin{aligned} P_{0,m} &= \{0\}, \\ P_{j+1,m} &= \bigcup_{a \in P_{j,m}} (a + N^{-(j+1)} P_m), \\ F_{j,m} &= \bigcup_{a \in P_{j,m}} a + [0, N^{-j}). \end{aligned}$$

Note that $F_{j,m} \subseteq E_{j,m}$ are smaller Cantor-type sets with endpoints in a rescaled arithmetic progression.

We now restate Theorem 6.2.7 in more details. Its proof is a routine modification of the proof of Theorem 6.2.6.

Theorem 6.2.7 (in detail). *For each $m = 1, \dots, k$ let $0 < \alpha_k < \dots < \alpha_1 < 1$ such that $\alpha_m = \frac{\log t_{0,m}}{\log n_0}$ for some $t_{0,m}, n_0 \in \mathbb{N}$, and $(k-1)\alpha_1 + \alpha_k < 2$. Let μ_m be the measures defined above, where the corresponding arithmetic progressions $\{P_m\}_{m=1}^k$ are $\left(\left\lceil \frac{1}{\sum_{m=1}^k \alpha_m} \right\rceil (N_0^{n_0 \alpha_1} - 1) + 1\right)$ -LI. Let $\{f_{\ell,m}\}_{\ell \in \mathbb{N}} = \{1_{F_{\ell,m}}\}_{\ell \in \mathbb{N}}$. Assume that $p, q \in [1, \infty]$ satisfy*

$$q < \frac{p(2 - \sum_{m=1}^k \alpha_m)}{(p-1) \sum_{m=1}^k \alpha_m}.$$

Then

$$\frac{\left\| \prod_{m=1}^k \widehat{f_{\ell,m} \mu_m} \right\|_{L^q(\mathbb{R})}}{\prod_{m=1}^k \|f_{\ell,m}\|_{L^p(\mu_m)}} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$

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