Decoupling for the parabola and the Kakeya conjecture

Ana E. de Orellana
Analysis Reading Group

Contents

Most of this material is based on [Gut17], [Gut22], and [ZK19]. Every sentence in this note should be taken with a pinch of salt, the objective here is not to be mathematically precise, please refer to the references for the correct statements.

1. Fourier analysis

Given a function $f \in L^1(\mathbb{R}^d)$, its Fourier transform is

$$\widehat{f}(\xi) = \int e^{-2\pi i \xi \cdot x} f(x)\, dx.$$

This definition can be extended to functions $f \in L^p(\mathbb{R}^d)$ by using the density of the Schwartz class (for $p = 2$) and by considering the functions f as tempered distributions (for any other value of $p \geqslant 1$). The inverse Fourier transform of f is

$$\widecheck{f}(x) = \int e^{2\pi i \xi \cdot x} f(\xi)\, d\xi,$$

or equivalently, $\widecheck{f}(x) = \widehat{f(-x)}$.

Fourier transforms behave nicely under linear transforms and differential operators, for that reason they usually appear in connection to differential equations, as we shall see later.

The following are some properties of Fourier transforms that will be useful to us. These need to be interpreted accordingly depending on where the function f lies. E.g. if f is in one of $\mathscr{S}(\mathbb{R}^d), L^1(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)$, everything is well-defined as stated, but if $f \in L^p(\mathbb{R}^d)$, then some of these identities should be interpreted as true in the distributional sense.

- Inversion formula: $\widecheck{\widehat{f}} = f = \widehat{\widecheck{f}}$;
- Parseval's formula: $\int f(x)\overline{g}(x)\, dx = \int \widehat{f}(\xi)\overline{\widehat{h}(\xi)}\, d\xi$;
- Plancherel's identity: $\|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|\widecheck{f}\|_{L^2(\mathbb{R}^d)}$;
- Hausdorff–Young inequality: $\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)}$, where p and p' are conjugate exponents.

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2. Decoupling

One of the most basic questions that we can ask in Fourier analysis is: What information can we get about f given information about $\widehat{f}$? Decoupling is one variant of this question, where "information about $\widehat{f}$ " refers to the geometry of the set where $\widehat{f}$ is supported, and the information that we want about f will be estimating $\|f\|_{L^p(\mathbb{R}^d)}$.

By the Fourier inversion formula we can write f as

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \widehat{f}(\xi)\, d\xi.$$

Suppose $\operatorname{spt} \widehat{f} \subseteq \Omega$ for some $\Omega \subseteq \mathbb{R}^d$, and that we can write Ω as the disjoint union $\Omega = \bigcup \theta$. Then

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \widehat{f}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^d} \sum_\theta e^{2\pi i \xi \cdot x} \widehat{f}|_\theta(\xi)\, d\xi \\ &= \sum_\theta \int_\theta e^{2\pi i \xi \cdot x} \widehat{f}(\xi)\, d\xi \\ &= \sum_\theta f_\theta, \end{aligned}$$

where $f_\theta := \int_\theta e^{2\pi i \xi \cdot x} \widehat{f}(\xi)\, d\xi$ is such that $\operatorname{spt} \widehat{f_\theta} \subseteq \theta$. So we are writing f as a sum of functions that depend on the support of $\widehat{f}$.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be the disjoint union $\Omega = \bigcup \theta$. For each $p \in [1, \infty]$, the decoupling constant $D_p(\Omega = \bigcup \theta)$ is the smallest constant such that for every f with $\operatorname{spt} \widehat{f} \subseteq \Omega$,*

$$\|f\|^2_{L^p(\mathbb{R}^d)} \leqslant D_p(\Omega = \textstyle\bigcup \theta)^2 \displaystyle\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^d)}.$$

Note that the decoupling constant depends both on Ω and its decomposition $\bigcup \theta$, as well as in the value of p.

There are some values of p that give trivial bounds for $D_p(\Omega = \bigcup \theta)$.

$\boldsymbol{p = 2}$: In this case the fact that θ are disjoint makes things trivial by orthogonality and the Pythagoras theorem,

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \sum_\theta \int_{\mathbb{R}^d} |f_\theta(x)|^2\, dx = \sum_\theta \|f_\theta\|^2_{L^2(\mathbb{R}^d)}.$$

Thus the decoupling constant is 1, and in this case it does not depend on the decomposition $\Omega = \bigcup \theta$.

$\boldsymbol{p = \infty}$: In this case we get a trivial upper bound for the decoupling constant.

$$\|f\|_{L^\infty(\mathbb{R}^d)} = \sup |f(x)| = \sup \Big| \sum_\theta f_\theta(x) \Big| \leqslant \sum_\theta \sup \|f_\theta\|_{L^\infty(\mathbb{R}^d)},$$

by Cauchy–Schwartz,

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leqslant \Big(\sum_\theta 1 \Big)^{1/2} \Big(\sum_\theta \|f_\theta\|^2_{L^\infty(\mathbb{R}^d)} \Big)^{1/2} = (\#\theta)^{1/2} \Big(\sum_\theta \|f_\theta\|^2_{L^\infty(\mathbb{R}^d)} \Big)^{1/2}.$$

Then $D_\infty(\Omega = \bigcup \theta) \leqslant \#\theta$, which clearly depends on the decomposition on Ω.

3. Decoupling for the parabola and the Kakeya problem

The parabola is an interesting object to study in frequency space for many reasons, one of them being that if a function is a solution to the Schrödinger equation[1], then its Fourier transform is supported in a parabola[2]. Let $P = \{(t, t^2) : -1 \leqslant t \leqslant 1\}$ be the truncated parabola. Let $\delta > 0$ be a small scale, Ω be the δ-neighbourhood of P, we will decompose Ω as the union of approximately rectangles θ of size $\delta^{1/2} \times \delta$.

Theorem 3.1. *For every $\varepsilon > 0$ and each $2 \leqslant p \leqslant 6$, $D_p(\Omega = \bigcup \theta) \lesssim \delta^{-\varepsilon}$.*

It was mentioned in Section 1 that the Fourier transform behaves nicely under linear transformations. Due to this, if we transform the vertical rectangles θ defined before by rectangles that are 'tangent' to the parabola, the decoupling constant will not change. So we will consider these other rectangles (or parallelepipeds) instead, that are still of size roughly $\delta^{-1/2} \times \delta$.

The proof of the decoupling theorem for the parabola uses the following tools

- *Orthogonality:* We saw that the functions f_θ are orthogonal on $\mathbb{R}^2$. They will be 'approximately orthogonal' on any sufficiently large set.
- *Induction on scales:* The definition of the decoupling constant was conveniently chosen so that it could combine information from different scales. In the case of the parabola, if we have two decompositions of Ω into rectangles at scales δ_1 and δ_2 respectively, then $D_p(\delta_1\delta_2) \leqslant D_p(\delta_1)D_p(\delta_2)$, where $D_p(\delta)$ refers to the decoupling constant for the decomposition into rectangles at scale δ. This allows us to consider, on each step, finer coverings of the parabola which will make the decoupling constant smaller.

Fix a scale δ, take a rectangle θ of size $\delta^{1/2} \times \delta$ in the decomposition of Ω. Since spt $\widehat{f_\theta} \subset \theta$, then the uncertainty principle says that[3] f_θ will be approximately constant on the dual rectangle θ^* centred at ω_θ and of size $\delta^{-1/2} \times \delta^{-1}$. Let $\mathbb{T}_\theta$ be a tiling of the plane by translations of θ^* (note that we changed the notation from rectangles to tubes because θ^*s are long and thin), we can write the wavepacket decomposition of f_θ,

$$f_\theta(x) \approx \sum_{T \in \mathbb{T}_\theta} a_T e^{2\pi i \omega_\theta \cdot x} \chi_T,$$

where χ_T is the characteristic function on T and a_T is some constant that depends on the tube T.

We have the same for each rectangle θ in the decomposition of Ω. By the curvature of the parabola, the rectangles θ will have different orientations, and so will their duals θ^*. This reduces the decoupling problem for the parabola to studying the overlaps of wavepackets with varying orientations, and thus, to study how much overlap there is between the long and thin tubes $T \in \mathbb{T}_\theta$ for different values of θ. The Kakeya conjecture asks precisely this question.

Conjecture 3.2. *For each θ in the covering of P, let T_θ be a translate of θ^*. Then for each $\varepsilon > 0$,*

$$\Big|\bigcup_\theta T_\theta\Big| \gtrsim \delta^\varepsilon \sum_\theta |T_\theta|.$$

[1] The Schrödinger equation is $\partial_t f = i\Delta f$.

[2] Other surfaces are also interesting for similar reasons, such as the sphere for Laplace eigenfunction equation $\Delta f = \lambda f$, and the cone for the wave equation $\partial_t^2 f = \Delta f$. In fact, decoupling was introduced by Wolff when studying local smoothing on the cone, a problem related to the solutions of the wave equation.

[3] Intuitively: If we know the frequency in concentrated in time then we don't have much certainty on the position of our particle, so the possible positions of the particle will be spread in space.

References

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A. E. de Orellana, University of St Andrews, Scotland

Email address: aedo1@st-andrews.ac.uk