Notes on the Brascamp–Lieb inequality

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Most of this material is based on [Dem20, BCT06, BCCT08, BCCT10]. Please refer to the references for more precise statements.

1. The Brascamp–Lieb inequality

The Brascamp–Lieb inequality was first introduced in [BL76] in an attempt by the authors to prove a sharp version of Young's inequality for convolutions. This inequality soon became an interesting object of study in the wider mathematical community, seeing applications in convex geometry, stochastic processes, and group theory, to name a few. In its most general form, the inequality first appeared in [Lie90] and states that

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(L_j x)\, dx \leqslant \mathrm{BL}(\boldsymbol{L}, \boldsymbol{p}) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^{n_j})} \tag{1.1}$$

where for each $j = 1, \dots, m$, $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ is a surjective linear map, $p_j \in [1, \infty]$, and the Brascamp–Lieb constant $\mathrm{BL}(\boldsymbol{L}, \boldsymbol{p})$ is the smallest constant such that the inequality holds for all non-negative $f \in L^{p_j}(\mathbb{R}^{n_j})$. The Brascamp–Lieb inequality quantifies how much the linear maps L_j can concentrate, and hence why it played a fundamental role in obtaining multilinear Kakeya and multilinear Fourier extension estimates.

This inequality is a powerful generalisation of several basic inequalities in analysis (see Section 2). Including Hölder's inequality, Young's inequality for convolutions, and the Loomis–Whitney inequality. All of which are obtained after making a convenient choice of Brascamp–Lieb datum, as we shall refer to the pair $(\boldsymbol{L}, \boldsymbol{p}) = ((L_j)_{j=1}^m, (p_j)_{j=1}^m)$.

Of course, (1.1) is only meaningful as long as the Brascamp–Lieb constant is finite, and it was only in [BCCT10, BCCT08] that the conditions that guarantee this were found. In fact, these conditions are not only sufficient, but also necessary.

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Theorem 1.1. *The Brascamp–Lieb constant* $BL(\boldsymbol{L},\boldsymbol{p})$ *is finite if and only if*

$$n=\sum_{j=1}^{m}\frac{n_j}{p_j} \tag{1.2}$$

and

$$\dim(V)\leqslant\sum_{j=1}^{m}\frac{\dim(L_jV)}{p_j} \tag{1.3}$$

for all subspaces V *of* $\mathbb{R}^n$*.*

Condition (1.2) states that the integrand on the left-hand side of (1.1) is essentially an n-dimensional object. Condition (1.3) measures how much of a given vector space V, the map L_j 'can see'. If the inequality was reversed then it would mean that the maps L_j are compressing V too much, or that V is close to being $\ker(L_j)$, and so we would be able to build functions f_j concentrated on those subspaces that would break the Brascamp–Lieb inequality.

The work of Lieb in [Lie90] illustrated the important role that Gaussians play in these types of inequalities. He proved that to obtain the sharp value of the Brascamp–Lieb constant it suffices to test the inequality with centred Gaussians, reducing the problem to a finite dimensional one.

2. Obtaining other fundamental inequalities

As mentioned previously, the Brascamp–Lieb inequality generalises several fundamental inequalties in analysis. In each of the following examples we use (1.2) and (1.3) to give the condition that guarantees the finiteness of the Brascamp–Lieb constant. Note, however, that finding that the value of such constant is equal to 1 is a much more laborious task, and comes from the fact that equality holds in all three inequalities when testing with appropriate Gaussians.

2.1. **Hölder's inequality.** For each $j=1,\dots,m$ let $n_j=n$, $L_j=\mathrm{Id}$. With these definitions the Brascamp–Lieb inequality becomes

$$\int_{\mathbb{R}^n}\prod_{j=1}^{m}f_j(x)\,dx\leqslant \mathrm{BL}(\mathbf{Id},\boldsymbol{p})\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

Conditions (1.2) and (1.3) hold trivially if the exponents p_j satisfy

$$\sum_{j=1}^{m}\frac{1}{p_j}=1$$

which gives rise to Hölder's inequality after proving that $\mathrm{BL}(\mathbf{Id})=1$.

2.2. **Young's inequality for convolutions.** Young's inequality states that for $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$,

$$\|f*g\|_{L^r(\mathbb{R}^n)}\leqslant\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}. \tag{2.1}$$

We will start by proving an equivalent representation of (2.1). For any $r\in[1,\infty]$ it holds that

$$\left|\int_{\mathbb{R}^n}h(x)p(x)\,dx\right|\leqslant M\|h\|_{L^{r'}(\mathbb{R}^n)}\quad\text{if and only if}\quad\|p\|_{L^r(\mathbb{R}^n)}\leqslant M, \tag{2.2}$$

where r' is the conjugate exponent of r.

Suppose we had that for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r'}=2$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}h(x)f(x-y)g(y)\,dy\,dx\leqslant\|h\|_{L^{r'}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}. \tag{2.3}$$

Since

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)f(x-y)g(y)\,dy\,dx = \int_{\mathbb{R}^n} h(x)(f*g)(x)\,dx,$$

letting $M = \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$ and $p = f*g$, (2.2) yields that

$$\|f*g\|_{L^r(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

The other direction is proved in the same way. With this, it suffices to prove (2.3), which we will using the Brascamp–Lieb inequality.

Let $n_1 = n_2 = n_3 = n$, let $p, q, r' \in [1,\infty]$, and define the transformations $L_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$L_1(x,y) = x, \quad L_2(x,y) = x-y, \quad L_3(x,y) = y.$$

Then the Brascamp–Lieb inequality reads

$$\int_{\mathbb{R}^{2n}} h(x)f(x-y)g(y)\,dx\,dy \leqslant \mathrm{BL}(\boldsymbol{L},\boldsymbol{p})\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\|h\|_{L^{r'}(\mathbb{R}^n)}$$

and $\mathrm{BL}(\boldsymbol{L},\boldsymbol{p})$ is finite (and equal to 1) if and only if

$$2 = \frac{1}{p} + \frac{1}{q} + \frac{1}{r'}, \tag{2.4}$$

thus proving (2.3). Note that (2.4) together with $\frac{1}{r} + \frac{1}{r'} = 1$ gives $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, which is the condition for the classic version of Young's inequality for convolutions.

2.3. **Loomis–Whitney inequality.** Let $(e_j)_{j=1}^n$ be the standard basis of $\mathbb{R}^n$. For each $j = 1,\dots,n$ let $n_j = n-1$, and $P_j : \mathbb{R}^n \to e_j^\perp$ be the orthogonal projection onto the complement of e_j. The Brascamp–Lieb inequality asserts that

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(x_1,\dots,\widehat{x_j},\dots,x_n)\,dx \leqslant \mathrm{BL}(\boldsymbol{P},\boldsymbol{n-1}) \prod_{j=1}^n \|f_j\|_{L^{n-1}(\mathbb{R}^{n-1})}, \tag{2.5}$$

where $\widehat{x_j}$ denotes omission. The Brascamp–Lieb constant is finite (and in fact, equal to 1) since

$$n = \sum_{j=1}^n \frac{n-1}{n-1},$$

and

$$\dim(V) \leqslant \sum_{j=1}^n \frac{\dim(P_j(V))}{n-1},$$

which follows easy from the rank-nullity theorem by checking several cases for the subspaces V.

Choosing for each $j = 1,\dots,n$, $f_j = \chi_{P_j(A)}$ for some set $A \subset \mathbb{R}^n$ gives a particularly interesting application of the Loomis–Whitney inequality,

$$|A| \leqslant |P_1(A)|^{\frac{1}{n-1}}\,|P_2(A)|^{\frac{1}{n-1}}\cdots|P_n(A)|^{\frac{1}{n-1}}.$$

3. Multilinear Kakeya and the Loomis–Whitney inequality

As we saw in Section 2.3, the Loomis–Whitney inequality is obtained from the Brascamp–Lieb inequality by making a specific choice of Brascamp–Lieb datum. Renaming the functions f_j in (2.5) by $f_j^{\frac{1}{n-1}}$ we obtain the following alternative version of the Loomis–Whitney inequality[1]

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(x_1, \dots, \widehat{x_j}, \dots, x_n)^{\frac{1}{n-1}}\, dx \leqslant \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}.$$

For each $j = 1, \dots, n$, let $\mathbb{T}_j$ be a family of infinitely long tubes of radius 1 *all* pointing in the x_j direction, and define $f_j = \sum_{T \in \mathbb{T}_j} \chi_T$. Then the Loomis–Whitney inequality for such functions reads

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} \left(\sum_{j=1}^{n} \chi_T \right)^{\frac{1}{n-1}} dx \leqslant \prod_{j=1}^{n} |\mathbb{T}_j|^{\frac{1}{n-1}}. \tag{3.1}$$

Thus, the Loomis–Whitney inequality allows us to obtain good control on the overlapping of families of *parallel* tubes pointing in transversal directions.

Now let us modify the family of tubes slightly. For each $j = 1, \dots, n$, let $\mathbb{T}_j$ be a family of infinitely long tubes of radius 1 pointing in directions belonging to some *sufficiently small fixed neighbourhood* of the x_j direction. The multilinear Kakeya estimate states that for $\frac{n}{n-1} \leqslant q \leqslant \infty$ the following inequality holds.

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} \left(\sum_{T \in \mathbb{T}_j} \chi_T \right)^{\frac{q}{n}} dx \lesssim \prod_{j=1}^{n} |\mathbb{T}_j|^{\frac{q}{n}}. \tag{3.2}$$

It is easy to see common ground here between (3.1) and (3.2) at the endpoint $q = \frac{n}{n-1}$. What is more, some sharp examples for the multilinear extension estimate also work for (3.1), which would indicate that these two estimates are equivalent. This is, of course, not the case: there are other examples for the multilinear Fourier extension estimate that do not work for (3.1), and in those examples the wavepackets of the functions f_j are supported on tubes pointing on directions that vary slightly; Just like in the tubes that form the families $\mathbb{T}_j$ in the multilinear Kakeya estimate. This is essentially the difference between (3.1) and (3.2), and the motivation behind the breakthrough paper [BCT06].

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[1]This modified version is also commonly seen for the Brascamp–Lieb inequality (1.1), see for example [BCCT08].

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